Computing the Tropical $\mathcal{A}$-discriminant

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Agenda

1. Project
2. Algorithm for n-variate $(n + 4)$-nomials
3. Future Work

Project

$\mathcal{A}$-discriminant Variety: Let $\mathcal{A} = \{a_1, a_2, \ldots, a_t\} \subseteq \mathbb{Z}^n$.
$\nabla_{\mathcal{A}}$ is the closure of

$$\left\{(c_1, \ldots, c_t) \in (\mathbb{C}^*)^n : f(x) = \sum_{i=1}^{t} c_i x^{a_i} \textit{ has a degenerate root}\right\}$$

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$\star$ We are interested in n-variate $(n+4)$-nomials

- Look at the connected components of $\mathbb{R}^t \setminus \nabla_{\mathcal{A}}$ called the **chambers**
- Visualize the topology of positive zero set of polynomials $\rightarrow$ Count the number of real roots $\rightarrow$ Approximate real roots

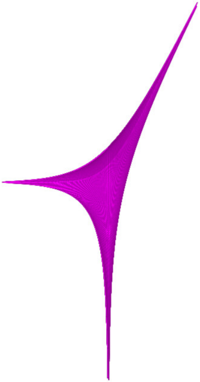
Zero set on Logarithmic paper

For any polynomial of the form $f(x) = \sum_{i=1}^{n} c_i x^{a_i}$,
Amoeba(f)$:= \{Log|x| \mid x_i \in \mathbb{C}^*, f(x) = 0\}$

Example: Let $f(x) = c_1 + c_2 x^{404} + c_3 x^{405} + c_4 x^{808}$
$\mathcal{A} = \{0, 404, 405, 808\}$

$\star$ $\mathcal{A}$-discriminant, $\Delta_{\mathcal{A}}$ has 609 monomial terms and degree 1604

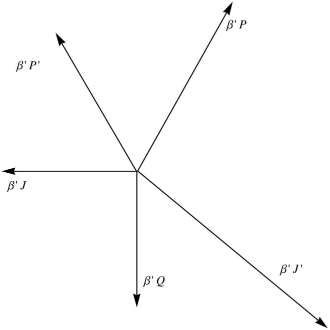
$\star$ Amoeba$(\Delta_{\mathcal{A}})$ = Log$|\nabla_{\mathcal{A}}|$ is a discriminant amoeba, and can be parametrized easily via the Horn-Kapranov Uniformization

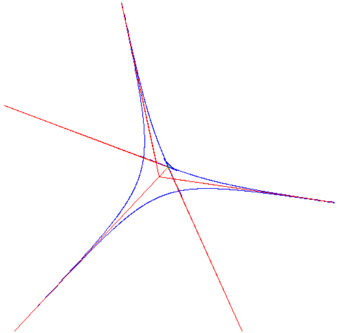


Tropical $\mathcal{A}$-Discriminant

⋆ Piecewise-linear polyhedral approximation of Amoeba($\Delta_{\mathcal{A}}$)
⋆ Gives us computationally tractable approximation of the discriminant chambers

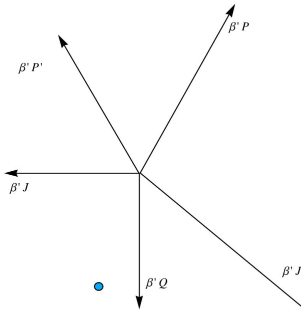
Example: Let $f(x, y) = c_0 + c_1 x + c_2 y + c_3 x^4 y + c_4 x y^4$ be a $(n+3)$-nomial for $c \in (\mathbb{C} \setminus \{0\})^{n+3}$.

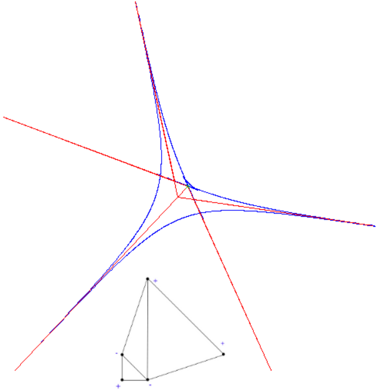




Tropical $\mathcal{A}$-discriminant

⋆ Provide results on the topology of real zero sets and faster homotopies preserving the number of real roots via the GKZ-correspondence





Project

⋆ For n-variate t-nomials:

Tropical $\Delta_{\mathcal{A}} \in \mathbb{R}^t$ approximates $\rightarrow$ Amoeba$(\Delta_{\mathcal{A}}) \in \mathbb{R}^t$

⋆ After Reduction:

Tropical $\overline{\Delta}_{\mathcal{A}} \in \mathbb{R}^{t-n-1}$ approximates
$\rightarrow$ Amoeba$(\overline{\Delta}_{\mathcal{A}}) \in \mathbb{R}^{t-n-1}$

Example:
⋆ For 1-variate $(n+4)$-nomials:

Tropical $\overline{\Delta}_{\mathcal{A}} \in \mathbb{R}^3$ approximates $\rightarrow$ Amoeba$(\overline{\Delta}_{\mathcal{A}}) \in \mathbb{R}^3$

Algorithm for n-variate $(n+4)$-nomials

Input: $\mathcal{A} \subset \mathbb{Z}^n$ of cardinality $n+4$

Output: Tropical $\mathcal{A}$-discriminant, $\tau(X_{\mathcal{A}}^*)$

1. Find the basis for the right null space B corresponding to $\hat{\mathcal{A}}$
2. Compute the intersections of the $-\beta_i$'s to find the vertices in $\mathcal{H}_B$
3. Take the linear combination of the $-\beta_i$'s to find the cones
4. Compute the 2-dimension cones that make up the walls corresponding to vertices of $\mathcal{H}_B$
5. The tropical $\mathcal{A}$-discriminant is the union of the walls

1-variate $(n+4)$-nomial

Let $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$.

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We need.. $\hat{\mathcal{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$

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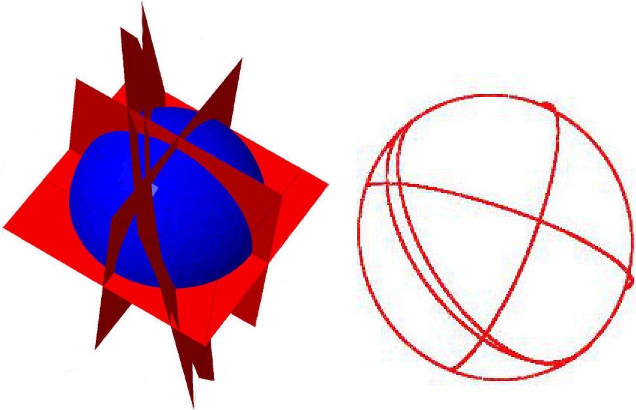
We find the basis for the right null space B corresponding to $\hat{\mathcal{A}}$

$$\text{B}= \begin{bmatrix} 3 & 2 & 1 \\ -4 & -3 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

1-variate $(n+4)$-nomial

Let $\mathcal{H}_B = \{[\lambda] \in \mathbb{P}_{\mathbb{C}}^{t-n-2} \mid \lambda \cdot \beta_i = 0 \textit{ for some } i \in \{1, ..., t\}\}$.

$\star$ When λ approaches the line corresponding to β_i and H-K-U blows up in the direction of $-\beta_i$, which are the **rays**



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$$B = \begin{bmatrix} 3 & 2 & 1 \\ -4 & -3 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} -\beta_1 &= (-3, -2, -1) \\ -\beta_2 &= (4, 3, 2) \\ -\beta_3 &= (0, 0, -1) \\ -\beta_4 &= (0, -1, 0) \\ -\beta_5 &= (-1, 0, 0) \end{aligned}$$

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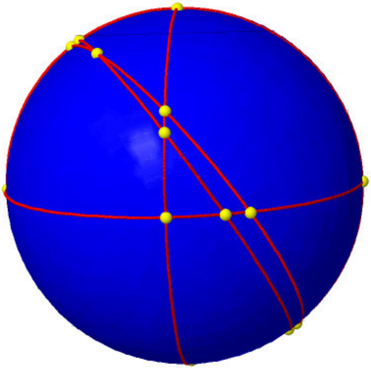
$\star$ Let W_v denote the cone generated by all $-\beta_i$ and β_i is normal to a hyperplane of $\mathcal{H}_B$ incident to the vertices, v.

$\star$ We call W_v a **wall** of $\mathcal{A}$.



1-variate $(n+4)$-nomial

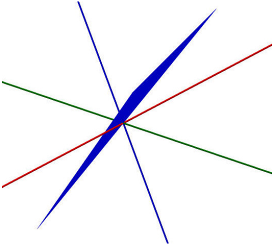
$\star$ Each wall is a 2-dimensional cone

Example:

$\beta_1 = (3,2,1)$, $\beta_2 = (-4,-3,-2)$

$\star$ Vertex of $-\beta_1, -\beta_2$ is $(1,-2,1)$ where the linear combinations of $-\beta_1, -\beta_2$ make up the cone generated by β_1, β_2

$\star$ $\text{Cone}(-\beta_1, -\beta_2) = \text{Cone}((-3,-2,-1),(4,3,2)) = \{(-3,-2,-1)s + (4,3,2)t \mid s,t \geq 0\}$



Tropical Discriminant

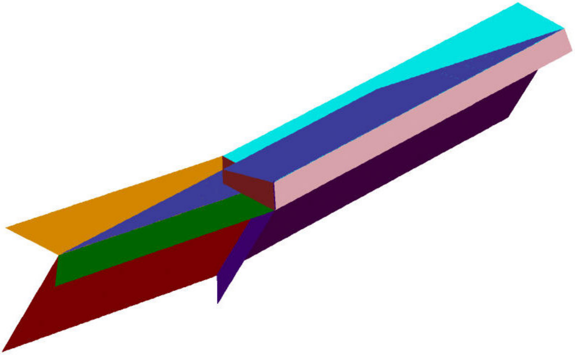
The **Tropical Discriminant** is the cone over the logarithmic limit set of $\Delta_{\mathcal{A}}$.

$\star$ We can look at $\nabla_{\mathcal{A}}$ and find its amoeba by taking the $Log|\cdot|$
$\star$ Then we can look at how the amoeba intersects a sphere
$\star$ The intersections yield a union of pieces of the great hemispheres in the limit as the radius goes to infinity
$\star$ If we connect the union of pieces to the origin we will get $\tau(X_{\mathcal{A}}^{*})$

1-variate $(n+4)$-nomial

Lemma 1.13 (Phillipson, Rojas)

The **Tropical Discriminant**, $\tau(X_{\mathcal{A}}^{*})$, is exactly the union of W_v over all vertices v of $\mathcal{H}_B$.



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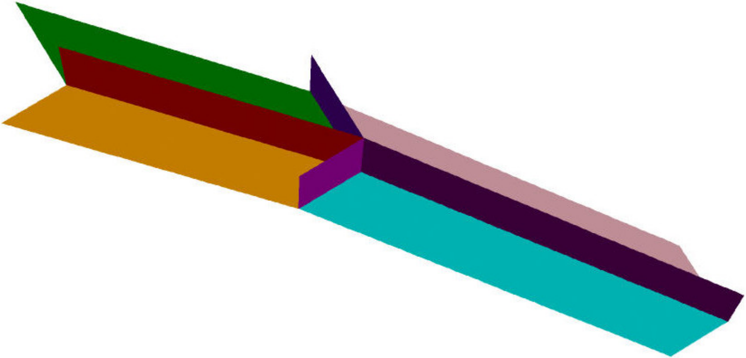

Movie

Future Work

⋆ We will develop a software package to quickly compute which $\mathcal{A}$-discriminant chamber contains the (n+4)-nomials

Input: $\mathcal{A} \subset \mathbb{Z}^n$ of cardinality $n + 4$ and the coefficient vector c of a given polynomial f

Output: Which chamber cone contain f

References

Bastani, Hillar, Popov & Rojas, 2011

Randomization, Sums of Squares, Near-Circuits, and Faster Real Root Counting

Contemporary Mathematics

Phillipson & Rojas

$\mathcal{A}$-Discriminants, and their Cuttings, for Complex Exponents

Thank you for listening!

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