THE CHOWLA-SELBERG FORMULA FOR QUARTIC ABELIAN CM FIELDS

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Abstract. We provide explicit analogues of the Chowla-Selberg formula for quartic abelian CM fields. This consists of two main parts. First, we implement an algorithm to compute the CM points at which we will evaluate a certain Hilbert modular function. Second, we exhibit families of quartic fields for which we can determine the precise form of the analogue of the product of gamma values.

1. Introduction and Statement of Results

The Chowla-Selberg formula [CSb], [CSa] relates the values of the Dedekind eta function $\eta(z)$ at certain CM points to values of Euler's gamma function $\Gamma(s)$ at rational numbers. The particular CM points appearing in the Chowla-Selberg formula are associated to imaginary quadratic fields. In [BSM], Barquero-Sanchez and Masri generalize the Chowla-Selberg formula to abelian CM fields. Their formula relates the values of a Hilbert modular function at CM points associated to an abelian CM field to values of both $\Gamma(s)$ and an analogous function $\Gamma_2 \colon \mathbb{R}^+ \to \mathbb{R}$ at rational numbers. Our goal is to provide explicit evaluations of the Chowla-Selberg formula for quartic abelian CM fields. This consists of two parts: (1) calculating the CM points, and (2) identifying families of field extensions for which the side of the formula involving $\Gamma(s)$ and $\Gamma_2(x)$ can be computed in general. We also provide several examples of our formulas for specific quartic fields.

We begin by explaining the Chowla-Selberg formula for abelian CM fields. Closely following the introduction in [BSM], let $F/\mathbb{Q}$ be a totally real field of degree n with complex embeddings $\tau_1, \ldots, \tau_n$. Let $\mathcal{O}_F$ be the ring of integers, $\mathcal{O}_F^\times$ be the group of units, d_F be the absolute value of the discriminant, ∂_F be the different, and

$$\zeta_F^*(s) = d_F^{s/2}\pi^{-ns/2}\Gamma(s/2)^n\zeta_F(s)$$

be the (completed) Dedekind zeta function. Let

$$z = x + iy = (z_1, \ldots, z_n) \in \mathbb{H}^n$$

where $\mathbb{H}$ is the complex upper half-plane. The Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_F)$ acts componentwise on $\mathbb{H}^n$ by linear fractional transformations. Define the real analytic function $\varphi \colon \mathbb{H}^n \to \mathbb{R}^+$ by

$$\varphi(z) = \exp\left(\frac{\zeta_F^*(-1)N(y)}{R_F} + \sum_{\substack{\mu \in \partial_F^{-1}/\mathcal{O}_F^\times \\ \mu \neq 0}} \frac{\sigma_1(\mu\partial_F)}{R_F\left|N_{F/\mathbb{Q}}(\mu\partial_F)\cdot N_{F/\mathbb{Q}}(\mu)\right|^{1/2}} e^{2\pi i T(\mu,z)}\right)$$

where $N(y) = \prod_{j=1}^n y_j$ is the product of the imaginary parts of the components of $z \in \mathbb{H}^n$, R_F is the residue of $\zeta_F^*(s)$ at $s = 0$,

$$\sigma_1(\mu\partial_F) = \sum_{\mathfrak{b}|\mu\partial_F} N_{F/\mathbb{Q}}(\mathfrak{b}),$$

and

$$T(\mu, z) = \sum_{j=1}^n \tau_j(\mu)x_j + i\sum_{j=1}^n |\tau_j(\mu)|y_j.$$

The function $\varphi(z)$ transforms like a weight one Hilbert modular form ([BSM], Lemma 2.1), and in the case $F = \mathbb{Q}$ we have $\varphi(z) = |\eta(z)|^2$. Now define the $\mathrm{SL}_2(\mathcal{O}_F)$-invariant function

$$H(z) = \sqrt{N(y)}\varphi(z).$$

Let E be a CM extension of F with class number h_E, and let $\Phi = (\sigma_1, \dots, \sigma_n)$ be a CM type for E. Assume that F has narrow class number 1. To define the CM points at which we will evaluate $H(z)$, note that given an ideal class $C \in \mathrm{CL}(E)$, there exists a fractional ideal $\mathfrak{a} \in C^{-1}$ such that

$$\mathfrak{a} = \mathcal{O}_F \alpha + \mathcal{O}_F \beta,$$

where $\alpha, \beta \in E^\times$ and

$$z_{\mathfrak{a}} = \frac{\beta}{\alpha} \in E^\times \cap \mathbb{H}^n = \{z \in E^\times \ : \ \Phi(z) = (\sigma_1(z), \dots, \sigma_n(z)) \in \mathbb{H}^n\}.$$

Then $z_{\mathfrak{a}}$ is a CM point corresponding to the inverse class $[\mathfrak{a}] = C^{-1}$. We will use the term CM point to refer to both $z_{\mathfrak{a}}$ and its image $\Phi(z_{\mathfrak{a}})$ under the CM type Φ, and it will be clear which point we mean either from context or notation. Let

$$\mathcal{CM}(E, \Phi, \mathcal{O}_F) = \{z_{\mathfrak{a}} \ : \ [\mathfrak{a}] \in \mathrm{CL}(E)\}$$

be a set of CM points of type (E, Φ). This is a CM 0-cycle on the Hilbert modular variety $\mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbb{H}^n$. In Section 4 we give more background on CM points and an algorithm to compute them when E is quartic abelian.

Let $L \subset \mathbb{Q}(\zeta_m)$ be an abelian field, where $\zeta_m = e^{2\pi i/m}$ is a primitive m-th root of unity. Let H_L be the subgroup of $G = \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ which fixes L. Recall that $G \cong (\mathbb{Z}/m\mathbb{Z})^\times$, where we identify the map $\sigma_t \in G$ determined by $\sigma_t(\zeta_m) = \zeta_m^t$ with the class $[t] \in (\mathbb{Z}/m\mathbb{Z})^\times$. We can then define the group of Dirichlet characters associated to L by

$$X_L = \{\chi \in (\widehat{\mathbb{Z}/m\mathbb{Z}})^\times \ : \ \chi\,|_{H_L} \equiv 1\}.$$

Since the CM field E is abelian over $\mathbb{Q}$, by the Kronecker-Weber theorem E embeds into some $\mathbb{Q}(\zeta_m)$. Hence $H_E \leq H_F$, and so $X_F \leq X_E$. For our purposes, the choice of cyclotomic field containing E does not matter. Note that we will frequently think of a character of a group $(\mathbb{Z}/N\mathbb{Z})^\times$ as a Dirichlet character without explicitly drawing the distinction.

Given a primitive Dirichlet character χ of conductor c_χ, let $L(\chi, s)$ be the Dirichlet L-function and let $\tau(\chi)$ be the Gauss sum

$$\tau(\chi) = \sum_{k=1}^{c_\chi} \chi(k) \zeta_{c_\chi}^k, \quad \zeta_{c_\chi} = e^{2\pi i/c_\chi}.$$

The gamma values in the Chowla-Selberg formula arise from Lerch's evaluation of $L'(\chi, 1)$ for χ an odd, primitive Dirichlet character. For their generalization to abelian CM fields, Barquero-Sanchez and Masri also had to evaluate $L'(\chi, 1)$ for χ an even, primitive Dirichlet character. Deninger [Den] showed how to evaluate $L'(\chi, 1)$ in terms of a certain function $\Gamma_2 \colon \mathbb{R}^+ \to \mathbb{R}$. To describe Γ_2, note that as a consequence of the Bohr-Mollerup theorem, the function $\log(\Gamma(x)/\sqrt{2\pi})$ is the unique function $f \colon \mathbb{R}^+ \to \mathbb{R}$ such that

$$f(x+1) - f(x) = \log(x),$$

$f(0) = \zeta'(0) = -\log\sqrt{2\pi}$, and $f(x)$ is convex on $\mathbb{R}^+$. Deninger proved that the function $-\zeta''(0, x)$ is the unique function $g \colon \mathbb{R}^+ \to \mathbb{R}$ such that

$$g(x+1) - g(x) = \log^2(x),$$

$g(1) = -\zeta''(0)$, and $g(x)$ is convex on (e, ∞). Here $\zeta(s)$ is the Riemann zeta function and

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad x > 0, \quad \mathrm{Re}(s) > 1$$

is the Hurwitz zeta function. Deninger also proved that

$$-\zeta''(0, x) = \lim_{n \to \infty} \left(\zeta''(0) + x \log^2(n) - \log^2(x) - \sum_{k=1}^{n-1} (\log^2(x+k) - \log^2(k)) \right).$$

Define

$$\Gamma_2(x) = \exp(-\zeta''(0, x)),$$

which by our above discussion can be viewed as analogous to $\Gamma(x)/\sqrt{2\pi}$. We can now state the version of the Chowla-Selberg formula for abelian CM fields.

Theorem 1.1 ([BSM], Theorem 1.1)**.** *Let $F/\mathbb{Q}$ be a totally real field of degree n with narrow class number 1. Let E/F be a CM extension with $E/\mathbb{Q}$ abelian. Let Φ be a CM type for E and*

$$\mathcal{CM}(E, \Phi, \mathcal{O}_F) = \{z_{\mathfrak{a}} \,:\, [\mathfrak{a}] \in \mathrm{CL}(E)\}$$

be a set of CM points of type (E, Φ). Then

$$\prod_{[\mathfrak{a}]\in\mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = c_1(E, F, n) \prod_{\chi\in X_E\setminus X_F} \prod_{k=1}^{c_\chi} \Gamma\left(\frac{k}{c_\chi}\right)^{\frac{h_E\chi(k)}{2L(\chi,0)}} \times \prod_{\substack{\chi\in X_F\\ \chi\neq 1}} \prod_{k=1}^{c_\chi} \Gamma_2\left(\frac{k}{c_\chi}\right)^{\frac{h_E\tau(\chi)\overline{\chi}(k)}{2c_\chi L(\chi,1)}}, \tag{1}$$

where

$$c_1(E, F, n) := \left(\frac{d_F}{2^{n+1}\pi\sqrt{d_E}}\right)^{\frac{h_E}{2}}.$$

Note that while X_F and X_E depend a priori on the choice of m in the embedding $E \subset \mathbb{Q}(\zeta_m)$, for the purposes of Theorem 1.1 we take the products on the right side of (1) over the primitive characters χ that induce the characters in X_E and X_F, with each primitive character appearing only once. This is the reason that (1) is independent of the choice of m in the embedding $E \subset \mathbb{Q}(\zeta_m)$. Equivalently, X_F and X_E may be taken to be the primitive Dirichlet characters associated to the Dirichlet L-functions in the factorizations of the Dedekind zeta functions of F and E, respectively.

Our main theoretical results are explicit evaluations of the right side of (1) in terms of standard arithmetical data associated to E and its subfields for a family of biquadratic CM fields and a family of cyclic quartic CM fields. We also describe and implement an algorithm to enumerate the CM points appearing on the left side of (1) for quartic abelian CM fields.

To state our results, for any squarefree integer d let $K = \mathbb{Q}(\sqrt{d})$. Let Δ be the discriminant of K, h_d be the class number, $w_d = \#\mathcal{O}_K^\times$ be the number of units (for $d < 0$), ϵ_d be a fundamental unit (for $d > 0$), and $\chi_d(k) = \left(\frac{\Delta}{k}\right)$ be the Kronecker symbol associated to K. Then we have the following evaluation of (1) for a family of biquadratic CM fields.

Theorem 1.2. *Let $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ be primes. Let $F = \mathbb{Q}(\sqrt{p}), E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$, and assume that F has narrow class number 1. Then*

$$\prod_{[\mathfrak{a}]\in\mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = \left(\frac{1}{8\pi q}\right)^{\frac{h_E}{2}} \prod_{k=1}^{q} \Gamma\left(\frac{k}{q}\right)^{\frac{h_E\chi_{-q}(k)w_{-q}}{4h_{-q}}} \prod_{k=1}^{pq} \Gamma\left(\frac{k}{pq}\right)^{\frac{h_E\chi_{-pq}(k)w_{-pq}}{4h_{-pq}}} \prod_{k=1}^{p} \Gamma_2\left(\frac{k}{p}\right)^{\frac{h_E\chi_p(k)}{4\log(\epsilon_p)}}. \tag{2}$$

For a Dirichlet character χ, let $B_1(\chi)$ be the first generalized Bernoulli number attached to χ. Our analogous result for a family of cyclic quartic CM fields is as follows.

Theorem 1.3. *Let $p \equiv 1 \pmod 4$ be a prime, and let $B, C > 0$ be integers such that $p = B^2+C^2$ and $B \equiv 2 \pmod 4$. Let $F = \mathbb{Q}(\sqrt{p})$ and $E = \mathbb{Q}\left(\sqrt{-\left(p + B\sqrt{p}\right)}\right)$, which is a cyclic quartic CM field with totally real quadratic subfield F. If F has narrow class number 1, then*

$$\prod_{[\mathfrak{a}]\in\mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = c_1(E, F, 2) \prod_{k=1}^{p} \Gamma\left(\frac{k}{p}\right)^{-h_E \operatorname{Re}\frac{\chi(k)}{B_1(\chi)}} \prod_{k=1}^{p} \Gamma_2\left(\frac{k}{p}\right)^{\frac{h_E\chi_p(k)}{4\log(\epsilon_p)}}. \tag{3}$$

Here χ is any choice of character of $(\mathbb{Z}/p\mathbb{Z})^\times$ that sends a primitive root modulo p to a primitive fourth root of unity.

Remark 1.4. The characters $\chi_p(k) = \left(\frac{k}{p}\right)$ and $\chi_{-q}(k) = \left(\frac{k}{q}\right)$ appearing in Theorem 1.2 are Legendre symbols, and χ_{-pq} is the character of $(\mathbb{Z}/pq\mathbb{Z})^\times$ equal to the product of the characters induced by χ_p and χ_{-q}. Because the Dirichlet character χ in Theorem 1.3 takes values in $\{0, \pm1, \pm i\}$, the exponents on the right side of (3) are rational except for the regulator $\log(\epsilon_p)$. Explicitly,

$$B_1(\chi) = \frac{1}{p}\sum_{k=1}^{p} \chi(k)(k+1).$$

This paper is organized as follows. In Section 2, we calculate the groups of characters appearing in the right side of Theorem 1.1 for the families of fields in Theorems 1.2 and 1.3. We also include a discussion of how to compute the group of characters for any cyclic quartic CM field. In Section 3, we prove Theorems 1.2 and 1.3. In Section 4, we review some relevant facts about CM 0-cycles on a Hilbert modular variety, and we give a modified version of an algorithm of Streng [Str10] to compute the CM points for a quartic abelian CM field. Finally, in Section 5 we give several examples of our main theorems and some tables of CM points for fields of small class number computed using the algorithm in Section 4. In the appendix we give our code in Sage [S$^+$13] for this algorithm.

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2. Groups of characters for quartic abelian CM fields

In this section we compute the groups of characters necessary to obtain Theorems 1.2 and 1.3. We also describe a more general method to compute the group of characters appearing on the right side of (1) for any cyclic quartic CM field.

First, we recall a couple of classical results about quadratic Gauss sums. For an odd prime p and integer k not divisible by p, let

$$g(k,p) = \sum_{x=1}^{p-1} \left(\frac{x}{p}\right) e^{2\pi i k x/p}$$

denote the quadratic Gauss sum. Then we have the following:

$$g(k,p) = \left(\frac{k}{p}\right) g(1,p), \qquad \text{([IR], Prop. 6.3.1)} \tag{4}$$

$$g(1,p) = \begin{cases} \sqrt{p}, & \text{if } p \equiv 1 \pmod 4, \\ \sqrt{-p}, & \text{if } p \equiv 3 \pmod 4, \end{cases} \qquad \text{([IR], Ch. 6, Theorem 1)} \tag{5}$$

where the square roots have positive real part. We can define a Gauss sum more generally by replacing the Legendre symbol with any character of $(\mathbb{Z}/p\mathbb{Z})^\times$, and then the analogue of (4) still holds. In ([BSM], Theorem 1.4), the authors give a more general version of Theorem 1.2 for multiquadratic extensions. Their computation of the group of characters for a multiquadratic extension relies on a certain Galois-theoretic property of the Kronecker symbol, which is more or less equivalent to the fact that an analogue of (5) holds when p is replaced by a squarefree integer (see [KKS], §5.2(d)-5.2(e)). In keeping with the concrete spirit of this paper, we give a direct computation of the group of characters for the family of biquadratic extensions in Theorem 1.2 using only (4)-(5).

Starting with the biquadratic case, let $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ be primes. Let $F = \mathbb{Q}(\sqrt{p})$, $E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$, and assume that F has narrow class number 1. Let $m = pq$. By (5) it is clear that $E \subset \mathbb{Q}(\zeta_m)$ where $\zeta_m = e^{2\pi i/m}$. Let $G = \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, and let H_F and H_E denote the subgroups that fix F and E, respectively. We would like to determine

$$X_F = \{\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times \;:\; \chi\,|_{H_F} = 1\}, \quad \text{and} \quad X_E = \{\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times \;:\; \chi\,|_{H_E} = 1\}.$$

Recall that we identify the automorphism $\sigma_t \in G$ determined by $\zeta_m \mapsto \zeta_m^t$ with the element $[t] \in (\mathbb{Z}/m\mathbb{Z})^\times$.

Lemma 2.1. *The automorphism $\sigma_t \in G$ fixes $F = \mathbb{Q}(\sqrt{p})$ if and only if t is a quadratic residue modulo p. Furthermore, σ_t fixes $E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$ if and only if t is a quadratic residue modulo p and t is a quadratic residue modulo q.*

Proof. This is a straightforward application of quadratic Gauss sums. Since $\zeta_p = \zeta_m^q$, then $\sigma_t(\zeta_p) = \zeta_p^t$. Hence by (4) and (5),

$$\sigma_t(\sqrt{p}) = \sigma_t(g(1,p)) = g(t,p) = \left(\frac{t}{p}\right)\sqrt{p}.$$

Thus σ_t fixes F if and only t is a quadratic residue modulo p. The same argument shows

$$\sigma_t(\sqrt{-q}) = \left(\frac{t}{q}\right)\sqrt{-q},$$

and the result follows. ∎

Now to determine X_F and X_E, note that

$$\widehat{(\mathbb{Z}/m\mathbb{Z})^\times} \cong \widehat{(\mathbb{Z}/p\mathbb{Z})^\times} \times \widehat{(\mathbb{Z}/q\mathbb{Z})^\times}.$$

We can choose this isomorphism so that we identify a pair of characters $\chi_p \in \widehat{(\mathbb{Z}/p\mathbb{Z})^\times}$ and $\chi_q \in \widehat{(\mathbb{Z}/q\mathbb{Z})^\times}$ with the product $\chi = \chi_p \cdot \chi_q$ of the characters of $(\mathbb{Z}/m\mathbb{Z})^\times$ that χ_p and χ_q induce.

Suppose $\chi = \chi_p \cdot \chi_q \in X_F$. By Lemma 2.1, $\chi(a) = 1$ if $\left(\dfrac{a}{p}\right) = 1$. Moreover, for any integer a such that $\left(\dfrac{a}{p}\right) = 1$, there exists a positive integer $b < pq$ such that $b \equiv a \pmod p$ and $b \equiv 1 \pmod q$. Since

$$\chi_p(a) = \chi_p(b)\chi_q(b) = \chi(b) = 1,$$

we see that χ_p is 1 on quadratic residues modulo p. Hence $\chi_p \equiv 1$ or $\chi_p(k) = \left(\dfrac{k}{p}\right)$. Similarly, for any integer a such that $\gcd(a, q) = 1$, there exists a positive integer $b < pq$ such that $b \equiv 1 \pmod p$ and $b \equiv a \pmod q$. Then

$$\chi_q(a) = \chi_p(b)\chi_q(b) = \chi(b) = 1,$$

so $\chi_q \equiv 1$. Hence we have shown that X_F is contained in the two element set consisting of trivial character and the character induced by $\left(\dfrac{k}{p}\right)$. It is straightforward to verify that this containment is in fact an equality.

Proceeding similarly to compute X_E, suppose $\chi = \chi_p \cdot \chi_q \in X_E$. By Lemma 2.1, $\chi(a) = 1$ if $\left(\dfrac{a}{p}\right) = 1$ and $\left(\dfrac{a}{q}\right) = 1$. By the exact same reasoning as in the first part of the previous paragraph, we deduce that $\chi_p \equiv 1$ or $\chi_p(k) = \left(\dfrac{k}{p}\right)$. By repeating this argument one more time, we find that $\chi_q \equiv 1$ or $\chi_q(k) = \left(\dfrac{k}{q}\right)$. Hence X_E is contained in the four element set consisting of the characters $\chi = \chi_p \cdot \chi_q$ where χ_p and χ_q are trivial or Legendre symbols. Again, a straightforward calculation shows that this containment is an equality. We have thus shown the following:

Lemma 2.2. *Let $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ be primes. Let $F = \mathbb{Q}(\sqrt{p}), E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$, and assume that F has narrow class number 1. Then X_F and X_E may be taken to be subsets of $(\mathbb{Z}/pq\mathbb{Z})^\times$, and*

$$X_F = \{\chi_1, \chi_2\}, \quad X_E = \{\chi_1, \chi_2, \chi_3, \chi_4\}$$

where

$$\chi_1 \equiv 1, \quad \chi_2(k) = \left(\frac{k}{p}\right), \quad \chi_3(k) = \left(\frac{k}{q}\right), \quad \chi_4(k) = \left(\frac{k}{p}\right) \cdot \left(\frac{k}{q}\right).$$

In the equations above, we mean the characters of $(\mathbb{Z}/pq\mathbb{Z})^\times$ induced by the Legendre symbols.

Moving on to the family of cyclic quartic CM fields in Theorem 1.3, we start with a general description of cyclic quartic fields following the introduction in [SW] and then specialize to our case. Let E be a cyclic quartic extension of $\mathbb{Q}$. Then there exist unique integers A, B, C, D such that

$$E = \mathbb{Q}\left(\sqrt{A(D + B\sqrt{D})}\right)$$

where

$$A \text{ is squarefree and odd,} \tag{6}$$

$$D = B^2 + C^2 \text{ is squarefree and } B, C > 0, \tag{7}$$

$$\gcd(A, D) = 1. \tag{8}$$

Moreover, each choice of integers A, B, C, D satisfying these conditions defines a cyclic quartic field with primitive element $\sqrt{A(D + B\sqrt{D})}$. The unique quadratic subfield is always $\mathbb{Q}(\sqrt{D})$.

It is now clear that the cyclic CM fields of degree 4 are precisely those cyclic extensions with $A < 0$. Indeed, if $A < 0$ then E is a totally imaginary quadratic extension of the totally real field $\mathbb{Q}(\sqrt{D})$, and if $A > 0$ then $A(D + B\sqrt{D})$ has a real embedding. Consider the following three cases:

$$\begin{cases} \text{Case 1:} & D \equiv 2 \pmod 4, \\ \text{Case 2:} & D \equiv 1 \pmod 4, \quad B \equiv 1 \pmod 2, \\ \text{Case 3:} & D \equiv 1 \pmod 4, \quad B \equiv 0 \pmod 2. \end{cases} \tag{9}$$

We further divide case 3 into the two subcases:

$$\begin{cases} \text{(a)} & A + B \equiv 3 \pmod 4, \\ \text{(b)} & A + B \equiv 1 \pmod 4, \end{cases} \tag{10}$$

Set

$$l = l(E) = \begin{cases} 3, & \text{in cases 1 and 2}, \\ 2, & \text{in case 3(a)}, \\ 0, & \text{in case 3(b)}. \end{cases} \tag{11}$$

We need to determine a cyclotomic field containing a given cyclic quartic CM field. The smallest such cyclotomic field possible is as follows.

Theorem 2.3 ([SW]). *Let $E = \mathbb{Q}\left(\sqrt{A(D + B\sqrt{D})}\right)$ be a cyclic quartic extension of $\mathbb{Q}$ where A, B, C, D are integers that satisfy (6) - (8). Let l be as in (11), and set $m = 2^l|A|D$. Then m is the least positive integer such that $E \subset \mathbb{Q}(\zeta_m)$.*

From this result, we can explicitly determine X_F and X_E for the family of cyclic quartic CM fields in Theorem 1.3 without doing any computations in F or E.

Lemma 2.4. *Let E be a cyclic quartic CM field of the form $E = \mathbb{Q}\left(\sqrt{-(p + B\sqrt{p})}\right)$ where $p \equiv 1 \pmod 4$ is prime and $p = B^2 + C^2$ for some integers $B, C > 0$ with $B \equiv 2 \pmod 4$. Let a be a primitive root modulo p. Then X_E and X_F may be taken as subsets of $(\mathbb{Z}/p\mathbb{Z})^\times$. More precisely,*

$$X_E = \{\chi_1, \chi_2, \chi_3, \overline{\chi}_3\}, \quad \text{and} \quad X_F = \{\chi_1, \chi_2\}$$

where

$$\chi_1 \equiv 1, \quad \chi_2(k) = \left(\frac{k}{p}\right), \quad \chi_3 \text{ is the character of } (\mathbb{Z}/p\mathbb{Z})^\times \text{ such that } \chi_3(a) = i.$$

Proof. By Theorem 2.3, the field E embeds into $\mathbb{Q}(\zeta_p)$ (this is case 3(b) as in (11)). Let $G = \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$. Since this group is cyclic, it is clear that there is a unique subgroup such that the corresponding quotient group is of order 4, namely, the subgroup $\langle a^4 \rangle$. Hence $H_E \cong \langle a^4 \rangle$, as $G/H_E \cong \mathrm{Gal}(E/\mathbb{Q})$. Similarly since H_F is a subgroup of G of index 2, then $H_F \cong \langle a^2 \rangle$.

Now let χ be a character of $(\mathbb{Z}/p\mathbb{Z})^\times$. In order for χ to be identically 1 on $\langle a^4 \rangle$, it is not hard to see that χ must be one of the four characters $\chi_1, \chi_2\chi_3, \overline{\chi}_3$. Clearly each of these characters is identically 1 on H_E, so this proves that $X_E = \{\chi_1, \chi_2, \chi_3, \overline{\chi}_3\}$. If χ is identically 1 on $H_F \cong \langle a^2 \rangle$, then χ is either χ_1 or χ_2, and both of these possibilities do occur so $X_F = \{\chi_1, \chi_2\}$. □

Remark 2.5. For every prime $p \equiv 1 \pmod 4$, the field $\mathbb{Q}(\zeta_p)$ contains a unique cyclic subfield E of order 4. Since p is prime, the field $\mathbb{Q}(\zeta_p)$ contains no other cyclotomic subfields. Hence $\mathbb{Q}(\zeta_p)$ is the smallest cyclotomic field containing E. By Lemma 2.3, E must be generated over $\mathbb{Q}$ by $\sqrt{A\left(p + B\sqrt{p}\right)}$ where $A = \pm 1$ and $p = B^2 + C^2$ for integers $B, C > 0$ with $B \equiv 0 \pmod 2$ and $A + B \equiv 1 \pmod 4$. If $A = -1$ then E is a CM field, otherwise E is not a CM field. The congruence $A + B \equiv 1 \pmod 4$ shows that A is easily obtained by knowing B. Additionally, B can be obtained from p since the decomposition $p = B^2 + C^2$ is unique up to order (assuming $B, C > 0$) and since precisely one of B, C is even in such a decomposition. Hence we have a simple congruence relation that determines when the cyclic quartic subfield of $\mathbb{Q}(\zeta_p)$ is a

CM field. Our family of fields in Theorem 1.3 is precisely the family of cyclic quartic CM fields with narrow class number 1 whose minimal cyclotomic embeddings are in the fields $\mathbb{Q}(\zeta_p)$. It is not known to the author whether this family of cyclic quartic CM fields is infinite (with or without the condition on the narrow class number). Computer experimentation suggests that when the fields $\mathbb{Q}(\zeta_p)$ with $p \equiv 1 \pmod 4$ are ordered by the size of p, on average half of these fields contain a cyclic quartic CM field.

We conclude this section by describing a general method for computing X_F and X_E for any cyclic quartic CM field, using [SW] as our primary reference. Recall the general setup. A cyclic quartic CM field E can be written uniquely as $E = \mathbb{Q}\left(\sqrt{A(D+B\sqrt{D})}\right)$ where A, B, C, D are integers satisfying (6) - (8) and $A < 0$. We classify E into one of the four cases according to (9)-(10), and we define l as in (11). By Theorem 2.3, the field E embeds into $\mathbb{Q}(\zeta_m)$ where $m = 2^l|A|D$. Hence we can view H_F and H_E as subgroups of $(\mathbb{Z}/m\mathbb{Z})^\times$.

Since $D = (\pm B)^2 + (\pm C)^2$ and $E = \mathbb{Q}\left(\sqrt{A(D \pm B\sqrt{D})}\right)$, we may change the signs of B and C without changing the field E. After replacing B by $-B$ and C by $-C$ if necessary, the number $\kappa \in \mathbb{Z}[i]$ defined by

$$\begin{cases} \text{Case 1:} & \kappa = \frac{1}{2}(B+C) + i\frac{1}{2}(C-B), \\ \text{Case 2:} & \kappa = B + iC, \\ \text{Case 3:} & \kappa = C + iB, \end{cases} \tag{12}$$

satisfies

$$\kappa \equiv 1 \pmod{(1+i)^3}$$

in the ring $\mathbb{Z}[i]$, that is, κ is primary. There exists a system of congruence relations on B, C, D that determine when to replace B by $-B$ and C by $-C$ explicitly described in [SW]. See [IR] for the definition of primary and other facts related to biquadratic reciprocity, which we will use below. We have

$$N(\kappa) = \kappa\overline{\kappa} = \begin{cases} \frac{1}{2}D, & \text{in case 1,} \\ D, & \text{in cases 2 and 3.} \end{cases} \tag{13}$$

where $N(\kappa)$ is the usual norm in $\mathbb{Z}[i]$. Since $N(\kappa)$ is squarefree and odd, and κ is prime, then κ is the (possibly empty) product $\pi_1 \cdots \pi_s$ of primary Gaussian primes whose norms $p_1, \ldots, p_s$ are distinct rational primes each congruent to 1 modulo 4. The empty product is understood to be 1, and this only occurs when $D = 2$, in which case $B = C = \kappa = 1$. We denote by $S_t(\pi_j)$ the Gauss sum

$$S_t(\pi_j) = \sum_{x=1}^{p_j-1} \chi_{\pi_j}(x) e^{2\pi i t x/p_j}.$$

Here $\chi_{\pi_j}(x)$ is the biquadratic residue character of $x \pmod{\pi_j}$, defined by

$$\chi_{\pi_j}(x) = i^j$$

where $i \in \mathbb{Z}[i]$ is the imaginary unit and $0 \le j \le 3$ is such that

$$x^{\frac{p_j-1}{4}} \equiv i^j \pmod{\pi_j}.$$

Note that

$$S_t(\pi_j) = \chi_{\pi_j}(t) S_1(\pi_j). \tag{14}$$

Set

$$S = S(\kappa) = \prod_{j=1}^{s} S_1(\pi_j). \tag{15}$$

For a primitive element for E, we have the following lemma.

Lemma 2.6 ([SW], Lemma 2.2). *Let $E = \mathbb{Q}\left(\sqrt{A(D+B\sqrt{D})}\right)$ be a cyclic quartic extension of $\mathbb{Q}$ where A, B, C, D are integers that satisfy (6) - (8). The minimal polynomial of $\sqrt{A(D+B\sqrt{D})}$ is $X^4 - 2ADX^2 + A^2C^2D$, whose roots are $\pm\sqrt{A(D\pm B\sqrt{D})}$. In terms of $S = S(\kappa)$, the roots of this minimal polynomial are*

$$\begin{cases} \text{Case 1:} & \pm\sqrt{A}(\omega S + \overline{\omega}\overline{S}), & \pm i\sqrt{A}(\omega S - \overline{\omega}\overline{S}), \\ \text{Case 2:} & \pm\sqrt{A}(S + \overline{S})/\sqrt{2}, & \pm i\sqrt{A}(S - \overline{S})/\sqrt{2}, \\ \text{Case 3:} & \pm\frac{1}{2}\sqrt{A}((1+i)S + (1-i)\overline{S}), & \pm\frac{1}{2}i\sqrt{A}((1-i)S + (1+i)\overline{S}), \end{cases}$$

where $\omega = e^{2\pi i/16}$.

Let $G = \mathrm{Gal}(\zeta_m/\mathbb{Q})$, and let H_E and H_F be the subgroups that fix E and $F = \mathbb{Q}(\sqrt{D})$, respectively. To compute H_E and H_F as subgroups of $(\mathbb{Z}/m\mathbb{Z})^\times$, we need to understand how an element $\sigma_t \in G$ defined by $\sigma(\zeta_m) = \zeta_m^t$ acts on $\sqrt{D}$ and a primitive element for E. It is sufficient to take $m = 2^3|A|D$, perform the necessary calculations, and then restrict to the smallest m possible given by Theorem 2.3.

To determine $\sigma_t(\sqrt{D})$ if D is odd, we can write $\sqrt{D}$ as a product of Gauss sums and a unit in $\mathbb{Z}$ by applying (5) to each of the prime factors of D. Indeed, since $D \equiv 1 \pmod 4$, the prime factors of D congruent to 3 modulo 4 come in pairs, so this product of Gauss sums will be $\pm\sqrt{D}$. By (4), we can determine how σ_t acts on each of these factors. If D is even, we add the factor $\sqrt{2} = e^{\pi i/4} - e^{7\pi i/4}$, which is mapped to $\left(\frac{t}{2}\right)\sqrt{2}$ by σ_t. Now in either case, we have all the information we need to compute H_F.

To compute H_E, we can use the equations for the generators of E as in Lemma 2.6. By the previous paragraph and (14), we just need to explain how σ_t acts on i, ω, $\overline{\omega}$, $\overline{G}$, and $\sqrt{A}$. As $4 \mid m$, we know how σ_t acts on $i = \zeta_m^{2|A|D}$. If $16 \mid m$ it is straightforward to determine $\sigma_t(\omega)$. Otherwise, let $M = 2m$ and consider the automorphism τ_t in $\mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ such that $\tau_t(\zeta_M) = \zeta_M^t$. As $16 \mid M$, we can determine how σ_t acts on ω. Since σ_t is the restriction of τ_t to $\mathbb{Q}(\zeta_m)$, we can then determine how σ_t acts on ω. To deal with $\overline{\omega}$, note that in a CM field the complex conjugation automorphism commutes with all other automorphisms. Thus $\sigma_t(\overline{\omega}) = \overline{\sigma_t(\omega)}$. This also impies that $\sigma_t(\overline{S}) = \overline{\sigma_t(S)}$. Finally, to determine $\sigma_t(\sqrt{A})$ we can use the method as described in the previous paragraph to compute $\sigma_t(\sqrt{D})$ (recall that A is odd and squarefree). Now in all three cases we have enough information to determine if σ_t fixes a primitive element for E.

We have now described how to determine whether or not a given automorphism $\sigma_t \in G$ fixes F or fixes E, so we can compute H_F and H_E explicitly as subgroups of $(\mathbb{Z}/m\mathbb{Z})^\times$. The determination of X_F and X_E is now a straightforward computational task, as it is not hard to enumerate all m characters of $(\mathbb{Z}/m\mathbb{Z})^\times$ and check which characters restrict to the identity on H_F or on H_E. We sketch the details of the calculation of X_E and X_F for a cyclic field in Example 5.5.

3. Proof of Main Results

We are now in a position to evaluate the right side of (1) for the families of quartic CM fields under consideration.

Proof of Theorem 1.2. We have

$$X_F = \{\chi_1, \chi_2\} \quad \text{and} \quad X_E = \{\chi_1, \chi_2, \chi_3, \chi_4\}$$

where $\chi_1, \chi_2, \chi_3, \chi_4$ are as in Lemma 2.1. Recall that for the purposes of evaluating the right side of (1), we view all characters χ as primitive characters of $(\mathbb{Z}/c_\chi\mathbb{Z})^\times$, where c_χ is the conductor of χ. Clearly χ_2 has conductor p, χ_3 has conductor q, and χ_4 has conductor pq. Straightforward calculations show that χ_2 is the Kronecker symbol associated $\mathbb{Q}(\sqrt{p})$, χ_3 is the Kronecker symbol associated to $\mathbb{Q}(\sqrt{-q})$, and χ_4 is the Kronecker symbol associated to $\mathbb{Q}(\sqrt{-pq})$. The residue of the Dedekind zeta function of $\mathbb{Q}(\sqrt{p})$ at $s = 1$ is equal to $L(\chi_2, 1)$, so by the class number formula

$$L(\chi_2, 1) = \frac{2h_p \log \epsilon_p}{\sqrt{p}}. \tag{16}$$

Recall that $\tau(\chi_2) = \sqrt{p}$ by (5), and that $h_p = 1$ by the narrow class number one assumption. To evaluate $L(\chi_3, 0)$, let $\zeta_K^*(s)$ be the completed Dedekind zeta function of the quadratic field $\mathbb{Q}(\sqrt{-q})$. Combining the class number formula with the functional equation for $\zeta_K^*(s)$ gives

$$L(\chi_3, 0) = \frac{2h_{-q}}{w_{-q}}.$$

Similarly

$$L(\chi_4, 0) = \frac{2h_{-pq}}{w_{-pq}}.$$

Since $p \equiv 1 \pmod 4$ and $-q \equiv 1 \pmod 4$, then $d_F = p$ and $d_{-q} = q$, where d_{-q} is the absolute value of the discriminant of $\mathbb{Q}(\sqrt{-q})$. Furthermore, as E is the compositum of the fields F and $\mathbb{Q}(\sqrt{-q})$ which have relatively prime discriminants, we have $d_E = (d_p d_{-q})^2 = p^2q^2$ ([Lan], Ch. 3, Prop. 17). It is now straightforward to compute the right side of (1) in terms of the data given. $\square$

Proof of Theorem 1.3. In the notation of Lemma 2.4,

$$X_E = \{\chi_1, \chi_2, \chi_3, \overline{\chi}_3\} \quad \text{and} \quad X_F = \{\chi_1, \chi_2\}.$$

Hence in this case all characters appearing on the right side of (1) have conductor p. As in the proof of Theorem 1.2, the character χ_2 is the Kronecker symbol associated to the field $\mathbb{Q}(\sqrt{p})$, so $L(\chi_2, 1)$ is given by (16). We do not have as refined a result as the class number formula for quadratic fields with which to evaluate the L-functions $L(\chi_3, s)$ and $L(\overline{\chi}_3, s)$ at $s = 0$, but we can write these values in terms of generalized Bernoulli numbers ([IR], Prop. 16.6.2). We have

$$L(\chi_3, 0) = -B_1(\chi_3),$$

where $B_1(\chi_3)$ is the first generalized Bernoulli number attached to χ_3. Similarly, $L(\overline{\chi}_3, 0) = -B_1(\overline{\chi}_3) = -\overline{B_1(\chi_3)}$. We can now evaluate the right side of (1) and arrive at Theorem 1.3. $\square$

4. Algorithm to Compute CM Points

Before presenting an algorithm to compute the CM points appearing in (1), we give more detailed information on CM 0-cycles on a Hilbert modular variety, essentially following the exposition of Bruinier and Yang ([BY], Sec. 3) and specializing to the case F has narrow class number 1. Let F be a totally real number field of degree n with real embeddings $\tau_1, \ldots, \tau_n$ and assume that F has narrow class number 1. Then $\mathrm{SL}_2(\mathcal{O}_F)$ acts on $\mathbb{H}^n$ via

$$Mz = (\tau_1(M)z_1, \ldots, \tau_n(M)z_n).$$

The quotient space $X(\mathcal{O}_F) = \mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbb{H}^n$ is the open Hilbert modular variety associated to $\mathcal{O}_F$. The variety $X(\mathcal{O}_F)$ parametrizes isomorphism classes of principally polarized abelian varieties (A, ι) with real multiplication $\iota : \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$.

Let E be a totally imaginary quadratic extension of F and $\Phi = (\sigma_1, \ldots, \sigma_n)$ be a CM type for E. A point $z = (A, \iota) \in X(\mathcal{O}_F)$ is a CM point of type (E, Φ) if one of the following equivalent conditions holds:

(1) As a point $z \in \mathbb{H}^n$, there is a point $\tau \in E$ such that

$$\Phi(\tau) = (\sigma_1(\tau), \ldots, \sigma_n(\tau)) = z$$

and such that

$$\Lambda_\tau = \mathcal{O}_F + \mathcal{O}_F\tau$$

is a fractional ideal of E.

(2) There exists a pair (A, ι') that is a CM abelian variety of type (E, Φ) with complex multiplication $i' : \mathcal{O}_E \hookrightarrow \mathrm{End}(A)$ such that $i = i' \mid_{\mathcal{O}_F}$.

By [BY, Lemma 3.2] and the narrow class number 1 assumption, there is a bijection between $\mathrm{CL}(E)$ and the CM points of type (E, Φ) defined as follows: given an ideal class $C \in \mathrm{CL}(E)$, there exists a fractional ideal $\mathfrak{a} \in C^{-1}$ and $\alpha, \beta \in E^\times$ such that we have the decomposition

$$\mathfrak{a} = \mathcal{O}_F\alpha + \mathcal{O}_F\beta \tag{17}$$

and such that $z = \beta/\alpha \in E^\times \cap \mathbb{H}^n = \{z \in E^\times : \Phi(z) \in \mathbb{H}^n\}$. Then z represents a CM point in $X(\mathcal{O}_F)$ in the sense that $\mathbb{C}^n/\Lambda_z$ is a principally polarized abelian variety of type (E, Φ) with complex multiplication by $\mathcal{O}_E$, where $\Lambda_z = \mathcal{O}_F + \mathcal{O}_F z$. Conversely, every principally polarized abelian variety of type (E, Φ) with complex multiplication by $\mathcal{O}_E$ arises from a decomposition as in (17) for some $\mathfrak{a}$ in a unique fractional ideal class in $\mathrm{CL}(E)$. We denote the CM 0-cycle consisting of the set of CM points of type (E, Φ) by $\mathcal{CM}(E, \Phi, \mathcal{O}_F)$ and identify it with the set

$$\{z_{\mathfrak{a}} \in E^\times \cap \mathbb{H}^n \ : \ [\mathfrak{a}] \in \mathrm{CL}(E)\}$$

under the bijection we just described.

As we noted earlier, the CM points $\mathcal{CM}(E, \Phi, \mathcal{O}_F) \subset X(\mathcal{O}_F)$ parametrize isomorphism classes of principally polarized abelian varieties (A, ι) with complex multiplication $\iota : \mathcal{O}_E \hookrightarrow \mathrm{End}(A)$. In his thesis, Streng [Str10] gives a couple of algorithms for enumerating principally polarized abelian varieties admitting complex multiplication by a CM field. In particular, ([Str10], Algorithm 2.5, Appendix 2) produces a list of elements $z_i \in E^\times$ such that for every fractional ideal class $C \in \mathrm{CL}(E)$ and any choice of CM type Φ, there is some $\mathfrak{a} \in C^{-1}$ and z_i such that $\mathfrak{a} = \mathcal{O}_F + \mathcal{O}_F z_i$ and the element $\epsilon = (z_i - \overline{z}_i)^{-1}\delta^{-1}$ is such that $\Phi(\epsilon) \in E^\times \cap \mathbb{H}^n$, where δ is a generator of the different ideal of F. The points z_i produced by this algorithm give us the decomposition (17), so we simply need to modify the algorithm to select z_i such that $\Phi(z_i) \in E^\times \cap \mathbb{H}^n$ rather than $\Phi(\epsilon) \in E^\times \cap \mathbb{H}^n$. Below we give a modified version of Streng's algorithm, where the only differences are that we have specialized to the case of a quadratic totally real subfield with narrow class number 1, and we are using a different choice of CM point as mentioned in the previous sentence.

Algorithm 4.1

INPUT: A quartic abelian CM field $E = F(\sqrt{\Delta})$ and a CM type Φ where $\Delta \in F$, F is totally real of degree 2, and F has narrow class number 1.

OUTPUT: A complete set of representatives $z_{\mathfrak{a}} \in \mathcal{CM}(E, \Phi, \mathcal{O}_F)$.

(1) Compute a set of representatives of the ideal class group of E.
(2) Compute an integral basis of $\mathcal{O}_E$.
(3) Write each element in the integral basis of $\mathcal{O}_E$ in the form $x_i + y_i\sqrt{\Delta}$ where $x_i, y_i \in F$.
(4) Compute all elements $a \in \mathcal{O}_F$ up to multiplication by $(\mathcal{O}_F^\times)^2$ such that

$$|N_{F/\mathbb{Q}}(a)| \leq \sqrt{|N_{F/\mathbb{Q}}(\Delta)|}\, d_F \frac{6}{\pi^2}. \tag{18}$$

(5) For each a, compute a complete set of representatives T_a for $\mathcal{O}_F/(a)$.
(6) For each a, compute all $b \in T_a$ for which a divides $b^2 - \Delta$.
(7) For each pair (a, b) and each basis element $x_i + y_i\sqrt{\Delta}$ of $\mathcal{O}_E$, check if $y_i a \in \mathcal{O}_F, x_i \pm y_i b \in \mathcal{O}_F$, and $a^{-1}y_i(\Delta - b^2) \in \mathcal{O}_F$ hold. Remove the pair (a, b) from the list if one of these conditions is not satisfied.
(8) For each pair (a, b), let

$$z = \frac{\sqrt{\Delta} - b}{a}.$$

Check if $\Phi(z) \in E^\times \cap \mathbb{H}^2$ and if

$$\mathfrak{a} = \mathcal{O}_F + \mathcal{O}_F z$$

is a fractional ideal of E. If so, then z is a CM point corresponding to the fractional ideal class of $\mathfrak{a}$. Search through the pairs (a, b) until a CM point has been found for each fractional ideal class in E.

In the appendix we present an implementation of Algorithm 4.1 in Sage.

5. Examples

We conclude by giving several examples of Theorems 1.2 and 1.3, including explicit CM points obtained by Algorithm 4.1 and a cyclic quartic field not covered by either of these theorems. For the convenience of the reader, we have included a couple of tables of CM points for biquadratic and cyclic quartic CM fields of small class number.

Example 5.1. (Theorem 1.3, $p = 13, q = 3$) Let $E = \mathbb{Q}(\sqrt{13}, \sqrt{-3})$ be a biquadratic CM field with totally real subfield $F = \mathbb{Q}(\sqrt{13})$. Note that F has narrow class number 1. Then $h_E = 2, h_{-3} = 1, h_{-39} = 2, h_{13} = 2, w_{-3} = 6$, and $w_{-39} = 2$. A fundamental unit is $\epsilon_{13} = (\sqrt{13}+3)/2$. Let $\Phi = \{\mathrm{id}, \sigma\}$ be the CM type where id is the identity map and σ is the automorphism such that $\sigma(\sqrt{13}) = -\sqrt{13}$ and $\sigma(\sqrt{-3}) = \sqrt{-3}$. By our implementation of Algorithm 4.1, we find that

$$z_1 = \frac{\sqrt{-3}-\sqrt{13}}{2}, \quad z_2 = \frac{\sqrt{-3}-3\sqrt{13}}{\sqrt{13}+5}$$

is a complete set of CM points of type (E, Φ). By Theorem 1.2, we compute

$$H\left(\frac{\sqrt{-3}-\sqrt{13}}{2}, \frac{\sqrt{-3}+\sqrt{13}}{2}\right) H\left(\frac{\sqrt{-3}-3\sqrt{13}}{\sqrt{13}+5}, \frac{\sqrt{-3}+3\sqrt{13}}{-\sqrt{13}+5}\right)$$
$$= \frac{1}{24\pi}\left(\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right)^3 \left(\frac{\Gamma\left(\frac{1}{39}\right)\Gamma\left(\frac{2}{39}\right)\Gamma\left(\frac{4}{39}\right)\Gamma\left(\frac{5}{39}\right)\Gamma\left(\frac{8}{39}\right)\Gamma\left(\frac{10}{39}\right)\Gamma\left(\frac{11}{39}\right)\Gamma\left(\frac{16}{39}\right)\Gamma\left(\frac{20}{39}\right)\Gamma\left(\frac{22}{39}\right)\Gamma\left(\frac{25}{39}\right)\Gamma\left(\frac{32}{39}\right)}{\Gamma\left(\frac{7}{39}\right)\Gamma\left(\frac{14}{39}\right)\Gamma\left(\frac{17}{39}\right)\Gamma\left(\frac{19}{39}\right)\Gamma\left(\frac{23}{39}\right)\Gamma\left(\frac{28}{39}\right)\Gamma\left(\frac{29}{39}\right)\Gamma\left(\frac{31}{39}\right)\Gamma\left(\frac{34}{39}\right)\Gamma\left(\frac{35}{39}\right)\Gamma\left(\frac{37}{39}\right)\Gamma\left(\frac{38}{39}\right)}\right)^{\frac{1}{4}}$$
$$\times \left(\frac{\Gamma_2\left(\frac{1}{13}\right)\Gamma_2\left(\frac{3}{13}\right)\Gamma_2\left(\frac{4}{13}\right)\Gamma_2\left(\frac{9}{13}\right)\Gamma_2\left(\frac{10}{13}\right)\Gamma_2\left(\frac{12}{13}\right)}{\Gamma_2\left(\frac{2}{13}\right)\Gamma_2\left(\frac{5}{13}\right)\Gamma_2\left(\frac{6}{13}\right)\Gamma_2\left(\frac{7}{13}\right)\Gamma_2\left(\frac{8}{13}\right)\Gamma_2\left(\frac{11}{13}\right)}\right)^{\frac{1}{2\log\left(\frac{\sqrt{13}+3}{2}\right)}}.$$

Example 5.2. (Theorem 1.2, $p = 29, q = 31$) For an example involving a larger class number, consider the biquadratic CM field $E = \mathbb{Q}(\sqrt{29}, \sqrt{-31})$. Then E has real quadratic subfield $F = \mathbb{Q}(\sqrt{29})$ with narrow class number 1. Similarly to Example 5.1, we will use the CM type $\Phi = \{\mathrm{id}, \sigma\}$ where $\sigma(\sqrt{29}) = -\sqrt{29}$ and $\sigma(\sqrt{-31}) = \sqrt{-31}$. We have $h_E = 21, h_{-31} = 3, h_{-29\cdot 31} = 14, w_{-31} = 2, w_{-29\cdot 31} = 2$ and $\epsilon_{29} = (\sqrt{29}-5)/2$. Table 3 gives a complete set of CM points $z_{\mathfrak{a}} \in \mathcal{CM}(E, \Phi, \mathcal{O}_F)$. Then Theorem 1.2 gives

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = \left(\frac{1}{248\pi}\right)^{\frac{21}{2}} \prod_{k=1}^{31}\Gamma\left(\frac{k}{31}\right)^{\frac{7}{2}\chi_{31}(k)} \prod_{k=1}^{899}\Gamma\left(\frac{k}{899}\right)^{\frac{3}{4}\chi(k)} \prod_{k=1}^{29}\Gamma\left(\frac{k}{29}\right)^{\frac{21\chi_{29}(k)}{4\log\left(\frac{\sqrt{29}-5}{2}\right)}},$$

where $\chi_{29}(k) = \left(\frac{k}{29}\right), \chi_{31}(k) = \left(\frac{k}{31}\right)$, and $\chi = \chi_{29}\cdot\chi_{31}$ is the product of the characters induced by χ_{29} and χ_{31} on $(\mathbb{Z}/899\mathbb{Z})^\times$, or equivalently, χ is the Kronecker symbol associated to the field $\mathbb{Q}(\sqrt{-899})$.

Example 5.3. (Theorem 1.3, $p = 13, A = -1, B = 2$) Consider the real quadratic field $F = (\mathbb{Q}(\sqrt{13})$, which has narrow class number 1. Since $13 = 2^2 + 3^2$, then by taking $B = 2$ as in Theorem 1.3, the field $E = F(\sqrt{-(13+2\sqrt{13})})$ is a cyclic CM field of degree 4 over $\mathbb{Q}$. Let $\Phi = \{\mathrm{id}, \tau\}$ be the CM type consisting of the identity map and the map τ such that $\tau\left(\sqrt{-(13+2\sqrt{13})}\right) = \sqrt{-(13-2\sqrt{13})}$. The number 2 is a primitive root modulo 13. Let χ be the character of $(\mathbb{Z}/13\mathbb{Z})^\times$ such that $\chi(2) = i$. A simple calculation shows $B_1(\chi) = -1-i$. We have $h_E = 1$, and a fundamental unit is $\epsilon_{13} = (\sqrt{13}+3)/2$. Finally, using our implementation of Algorithm 4.1, we compute that $(\sqrt{-(13+2\sqrt{13})} - 3\sqrt{13})/(\sqrt{13}+5)$ is a CM point corresponding to the only fractional ideal class in E. We have $d_F = 13$ and $d_E = 2197$. Now by Theorem 1.3,

$$H\left(\frac{\sqrt{-(13+2\sqrt{13})}-3\sqrt{13}}{\sqrt{13}+5}, \frac{\sqrt{-(13-2\sqrt{13})}+3\sqrt{13}}{-\sqrt{13}+5}\right)$$
$$= \frac{1}{13^{\frac{1}{4}}\sqrt{2^3}\pi}\left(\frac{\Gamma\left(\frac{1}{13}\right)\Gamma\left(\frac{2}{13}\right)\Gamma\left(\frac{3}{13}\right)\Gamma\left(\frac{5}{13}\right)\Gamma\left(\frac{6}{13}\right)\Gamma\left(\frac{9}{13}\right)}{\Gamma\left(\frac{4}{13}\right)\Gamma\left(\frac{7}{13}\right)\Gamma\left(\frac{8}{13}\right)\Gamma\left(\frac{10}{13}\right)\Gamma\left(\frac{11}{13}\right)\Gamma\left(\frac{12}{13}\right)}\right)^{\frac{1}{2}}$$
$$\times \left(\frac{\Gamma_2\left(\frac{1}{13}\right)\Gamma_2\left(\frac{3}{13}\right)\Gamma_2\left(\frac{4}{13}\right)\Gamma_2\left(\frac{9}{13}\right)\Gamma_2\left(\frac{10}{13}\right)\Gamma_2\left(\frac{12}{13}\right)}{\Gamma_2\left(\frac{2}{13}\right)\Gamma_2\left(\frac{5}{13}\right)\Gamma_2\left(\frac{6}{13}\right)\Gamma_2\left(\frac{7}{13}\right)\Gamma_2\left(\frac{8}{13}\right)\Gamma_2\left(\frac{11}{13}\right)}\right)^{\frac{1}{4\log\left(\frac{\sqrt{13}+3}{2}\right)}}.$$

Example 5.4. (Theorem 1.3, $p = 109, A = -1, B = 10$) Let $F = \mathbb{Q}(\sqrt{109})$ and $E = \mathbb{Q}\left(\sqrt{-\left(109+10\sqrt{109}\right)}\right)$. Since $109 = 10^2 + 3^2$, then E is a cyclic quartic CM field with totally real quadratic subfield F. The class number of E is $h_E = 17$, and F has narrow class number 1. Let $\Phi = \{\mathrm{id}, \tau\}$ be the CM type consisting of the identity map and the map τ such that $\tau(\sqrt{-\left(109+10\sqrt{109}\right)}) = \sqrt{-\left(109-10\sqrt{109}\right)}$. The number 6 is a primitive root modulo 109, so let χ be the character of $(\mathbb{Z}/109\mathbb{Z})^\times$ such that $\chi(6)=i$. Then $B_1(\chi)=-5-3i$. A fundamental unit is $\epsilon_{109}=(25\sqrt{109}+261)/2$, and we have $d_F=109, d_E=109^3$. Substituting this data into Theorem 1.3, we get

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = \left(\frac{1}{8\pi\sqrt{109}}\right)^{\frac{17}{2}} = \prod_{k=1}^{108}\Gamma\left(\frac{k}{109}\right)^{\frac{1}{2}\alpha(k)} \prod_{k=1}^{108}\Gamma_2\left(\frac{k}{109}\right)^{\frac{17\chi_{109}(k)}{4\log\left(\frac{25\sqrt{109}+261}{2}\right)}},$$

where the 17 CM points are given in Table 4, $\chi_{109}(k)=\left(\frac{k}{109}\right)$, and

$$\alpha(k)=\begin{cases} 5, & \text{if } \chi(k)=1,\\ -5, & \text{if } \chi(k)=-1,\\ 3, & \text{if } \chi(k)=i,\\ -3, & \text{if } \chi(k)=-i.\end{cases}$$

Example 5.5. (D=5, A=-3, B=1) For an example of a cyclic quartic CM field not covered by Theorem 1.3, let $E=\mathbb{Q}\left(\sqrt{-3\left(5+\sqrt{5}\right)}\right)$ and $F=\mathbb{Q}(\sqrt{5})$, which has narrow class number 1. Then we are in case 2 as in (11), so let $m=2^3\cdot 3\cdot 5=120$. By Lemma 2.3, m is the least positive integer such that $E\subset\mathbb{Q}(\zeta_m)$. Let $G=\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, and denote by $\sigma_t\in G$ the automorphism such that $\sigma_t(\zeta_m)=\zeta_m^t$. As determined by (12), we have that $\kappa=-1+2i\in\mathbb{Z}[i]$ is primary. Moreover, κ is prime. Set $S=S_1(\kappa)$ as in (15). By Lemma 2.6, a primitive element for E over $\mathbb{Q}$ is

$$\sqrt{-3}(S+\overline{S})/\sqrt{2}.$$

By Lemma 2.2,

$$H_F=\left\{\sigma_t\in G\ :\ \left(\frac{t}{5}\right)=1\right\}.$$

A calculation shows that, as primitive characters,

$$X_F=\{\chi_1,\chi_2\}$$

where

$$\chi_1\equiv 1,\quad \chi_2(k)=\left(\frac{k}{5}\right).$$

To determine H_E, note that by the discussion at the end of Section 2,

$$\sigma_t(\sqrt{-3})=\left(\frac{t}{3}\right),\quad \chi_t(\sqrt{2})=\left(\frac{t}{2}\right),\quad \sigma_t(S)=\chi_\kappa(t),$$

where χ_κ is the biquadratic residue character modulo κ. Hence if

$$\left(\frac{t}{3}\right)\left(\frac{t}{2}\right)\chi_\kappa(t)=1, \tag{19}$$

then it is certainly true that $\sigma_t\in H_F$. A calculation shows that (19) holds for $t\in\{1,59,71,79,89,91,101,109\}$. This set has order 8, and we know that H_E has order 8 since G/H_F is cyclic of order 4 and $|G|=32$, so this set must be H_E. By another calculation we find that

$$X_F=\{\chi_1,\chi_2,\chi_3,\overline{\chi}_3\}$$

where χ_1 and χ_2 are as before, and χ_3 is the character of conductor 120 such that

$$\chi_3(31)=-1,\quad \chi_3(61)=-1,\quad \chi_3(41)=-1,\quad \chi_3(97)=i.$$

We have $d_F=5, d_E=2^6\cdot 3^2\cdot 5^3, h_E=4, B_1(\chi)=2+2i$, and $\epsilon_5=(\sqrt{5}+1)/2$. Let $\Phi=\{\mathrm{id},\tau\}$ be the CM type where $\tau\left(\sqrt{-3\left(5+\sqrt{5}\right)}\right)=\sqrt{-3\left(5-\sqrt{5}\right)}$. Proceeding in the same way as in the proof of Theorem 1.3, we find

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = \frac{1}{2^{12}\cdot 3^2\cdot 5\cdot \pi^2}\prod_{k=1}^{120}\Gamma\left(\frac{k}{120}\right)^{\alpha(k)}\left(\frac{\Gamma_2\left(\frac{1}{5}\right)\Gamma_2\left(\frac{4}{5}\right)}{\Gamma_2\left(\frac{2}{4}\right)\Gamma_2\left(\frac{3}{4}\right)}\right)^{\frac{1}{\log\left(\frac{1+\sqrt{5}}{2}\right)}},$$

where

$$\alpha(k) = \begin{cases} 1, & \text{if } \chi(k) = -1 \text{ or } \chi(k) = -i, \\ -1, & \text{if } \chi(k) = 1 \text{ or } \chi(k) = i, \\ 0, & \text{if } \gcd(k, 120) > 1, \end{cases}$$

and the four CM points $z_{\mathfrak{a}}$ computed using Algorithm 4.1 are

$$z_{\mathfrak{a}_1} = \sqrt{-3\left(5+\sqrt{5}\right)},\quad z_{\mathfrak{a}_2} = \frac{\sqrt{-3\left(5+\sqrt{5}\right)}}{2}, z_{\mathfrak{a}_3} = \frac{2\sqrt{-3\left(5+\sqrt{5}\right)}}{\sqrt{5}+5}, z_{\mathfrak{a}_4} = \frac{\sqrt{-3\left(5+\sqrt{5}\right)}}{3}$$

Table 1 shows a complete set of representatives of the CM points for the biquadratic CM fields $E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$ and all primes p, $q < 50$ such that $p \equiv 1 \pmod 4$, $q \equiv 3 \pmod 4$, the class number of E is less than 4, and F has narrow class number 1. Here the CM type is given by $\Phi = \{\mathrm{id}, \sigma\}$ where σ is the automorphism of E such that $\sigma(\sqrt{p}) = -\sqrt{p}$ and $\sigma(\sqrt{-q}) = \sqrt{-q}$. Table 2 shows a complete set of representatives of the CM points for the cyclic CM fields $E = \mathbb{Q}\left(\sqrt{-\left(p+B\sqrt{p}\right)}\right)$ where p, B are as in Theorem 1.3 and $p \leq 101$. The CM type is $\Phi = \{\mathrm{id}, \tau\}$ where $\tau\left(\sqrt{-\left(p+B\sqrt{p}\right)}\right) = \sqrt{-\left(p-B\sqrt{p}\right)}$. In both cases, the CM points z are such that each $\mathcal{O}_F + \mathcal{O}_F z$ belongs to a distinct fractional ideal class in E. For each field E, the first CM point listed corresponds to the fractional ideal class of $\mathcal{O}_E$.

Table 1: CM points for biquadratic fields $\mathbb{Q}(\sqrt{p}, \sqrt{-q})$ with small class number

Primes	$\#\,\mathrm{CL}(\mathbb{Q}(\sqrt{p}, \sqrt{-q}))$	CM Points		
$p = 5,\quad q = 3$	1	$\frac{\sqrt{-3}-\sqrt{5}}{2}$		
$p = 5,\quad q = 7$	1	$\frac{\sqrt{-7}-\sqrt{5}}{2}$		
$p = 5,\quad q = 11$	2	$\frac{\sqrt{-11}-\sqrt{5}}{2}$	$\frac{\sqrt{-11}+\sqrt{5}}{4}$	
$p = 5,\quad q = 23$	3	$\frac{\sqrt{-23}-\sqrt{5}}{2}$	$\frac{\sqrt{-23}-1}{4}$	$\frac{\sqrt{-23}-2\sqrt{5}-1}{4}$
$p = 13,\quad q = 3$	2	$\frac{\sqrt{-3}-\sqrt{13}}{2}$	$\frac{\sqrt{-3}-3\sqrt{13}}{\sqrt{13}+5}$	
$p = 13,\quad q = 7$	1	$\frac{\sqrt{-7}-\sqrt{13}}{2}$		
$p = 13,\quad q = 19$	3	$\frac{\sqrt{-19}-\sqrt{13}}{2}$	$\frac{\sqrt{-19}+\sqrt{13}}{4}$	$\frac{\sqrt{-19}-\sqrt{13}}{4}$
$p = 13,\quad q = 31$	3	$\frac{\sqrt{-31}-\sqrt{13}}{2}$	$\frac{\sqrt{-31}-1}{4}$	$\frac{\sqrt{-31}-2\sqrt{13}-1}{4}$
$p = 17,\quad q = 3$	1	$\frac{\sqrt{-3}-\sqrt{17}}{2}$		

$p = 17, \quad q = 11$	1	$\dfrac{\sqrt{-11} - \sqrt{17}}{2}$		
$p = 17, \quad q = 19$	2	$\dfrac{\sqrt{-19} - \sqrt{17}}{2}$	$\dfrac{\sqrt{-19} + \sqrt{17}}{6}$	
$p = 29, \quad q = 3$	3	$\dfrac{\sqrt{-3} - \sqrt{29}}{2}$	$\dfrac{\sqrt{-3} + \sqrt{29}}{4}$	$\dfrac{\sqrt{-3} - \sqrt{29}}{4}$
$p = 29, \quad q = 7$	2	$\dfrac{\sqrt{-7} - \sqrt{29}}{2}$	$\dfrac{\sqrt{-7} - 7\sqrt{29}}{-3\sqrt{29} + 17}$	
$p = 37, \quad q = 7$	2	$\dfrac{\sqrt{-7} - \sqrt{37}}{2}$	$\dfrac{\sqrt{-7} - 7\sqrt{37}}{-9\sqrt{37} + 55}$	
$p = 41, \quad q = 3$	1	$\dfrac{\sqrt{-3} - \sqrt{41}}{2}$		
$p = 41, \quad q = 11$	3	$\dfrac{\sqrt{-11} - \sqrt{41}}{2}$	$\dfrac{\sqrt{-11} + 3\sqrt{41}}{-124\sqrt{41} + 794}$	$\dfrac{\sqrt{-11} - 3\sqrt{41}}{-124\sqrt{41} + 794}$

Table 2: CM points for cyclic quartic fields $\mathbb{Q}\left(\sqrt{-\left(p + B\sqrt{p}\right)}\right) = \mathbb{Q}(\sqrt{\Delta})$ as in Theorem 1.3 for $p \leq 101$.

Parameters	$\#\,\mathrm{CL}\left(\mathbb{Q}\left(\sqrt{-\left(p + B\sqrt{p}\right)}\right)\right)$	CM Points		
$p = 5, \quad B = 2$	1	$\dfrac{\sqrt{\Delta} - \sqrt{5}}{2}$		
$p = 13, \quad B = 2$	1	$\dfrac{\sqrt{\Delta} - 3\sqrt{13}}{\sqrt{13} + 5}$		
$p = 29, \quad B = 2$	1	$\dfrac{\sqrt{\Delta} - 5\sqrt{29}}{-4\sqrt{29} + 22}$		
$p = 37, \quad B = 6$	1	$\dfrac{\sqrt{\Delta} - \sqrt{37}}{2}$		
$p = 53, \quad B = 2$	1	$\dfrac{\sqrt{\Delta} - 7\sqrt{53}}{\sqrt{53} + 9}$		
$p = 61, \quad B = 6$	1	$\dfrac{\sqrt{\Delta} - 5\sqrt{61}}{-\sqrt{61} + 9}$		
$p = 101, \quad B = 10$	5	$\dfrac{\sqrt{\Delta} - \sqrt{101}}{2}$	$\dfrac{\sqrt{\Delta} + 3\sqrt{101}}{19\sqrt{101} + 191}$	$\dfrac{\sqrt{\Delta} - 3\sqrt{101}}{19\sqrt{101} + 191}$
		$\dfrac{\sqrt{\Delta} + 3\sqrt{101}}{\sqrt{101} + 11}$	$\dfrac{\sqrt{\Delta} - 3\sqrt{101}}{\sqrt{101} + 11}$	

The biquadratic field $\mathbb{Q}(\sqrt{29}, \sqrt{-31})$ has class number 21 and the cyclic field $\mathbb{Q}\left(\sqrt{-\left(109+10\sqrt{109}\right)}\right)$ has class number 17. Tables 3 - 4 show CM points corresponding to the fractional ideal classes computed using the code in the appendix. The CM points and CM types are as in Tables 1 - 2.

Table 3: CM points for $\mathbb{Q}\left(\sqrt{29}, \sqrt{-31}\right)$ (see Example 5.2)

$\frac{\sqrt{-31}-\sqrt{29}}{2}$	$\frac{\sqrt{-31}-1}{4}$	$\frac{\sqrt{-31}-2\sqrt{29}-1}{4}$
$\frac{\sqrt{-31}+\sqrt{29}}{-\sqrt{29}+7}$	$\frac{\sqrt{-31}-\sqrt{29}}{-\sqrt{29}+7}$	$\frac{\sqrt{-31}+\sqrt{29}}{-4\sqrt{29}+22}$
$\frac{\sqrt{-31}-\sqrt{29}}{-4\sqrt{29}+22}$	$\frac{\sqrt{-31}+5\sqrt{29}}{-3\sqrt{29}+17}$	$\frac{\sqrt{-31}-5\sqrt{29}}{-3\sqrt{29}+17}$
$\frac{\sqrt{-31}+5\sqrt{29}}{-2\sqrt{29}+12}$	$\frac{\sqrt{-31}-5\sqrt{29}}{-2\sqrt{29}+12}$	$\frac{\sqrt{-31}+\sqrt{29}}{6}$
$\frac{\sqrt{-31}-\sqrt{29}}{6}$	$\frac{\sqrt{-31}+4\sqrt{29}-1}{-2\sqrt{29}+14}$	$\frac{\sqrt{-31}+2\sqrt{29}-1}{-2\sqrt{29}+14}$
$\frac{\sqrt{-31}+19\sqrt{29}}{-7\sqrt{29}+39}$	$\frac{\sqrt{-31}-19\sqrt{29}}{-7\sqrt{29}+39}$	$\frac{\sqrt{-31}+19\sqrt{29}}{-3\sqrt{29}+19}$
$\frac{\sqrt{-31}-19\sqrt{29}}{-3\sqrt{29}+19}$	$\frac{\sqrt{-31}+10\sqrt{29}-1}{-6\sqrt{29}+34}$	$\frac{\sqrt{-31}-8\sqrt{29}-1}{-6\sqrt{29}+34}$

Table 4: CM points for $\mathbb{Q}\left(\sqrt{-\left(109+10\sqrt{109}\right)}\right) = \mathbb{Q}(\sqrt{\Delta})$ (see Example 5.4)

$\frac{\sqrt{\Delta}-3\sqrt{109}}{-\sqrt{109}+11}$	$\frac{\sqrt{\Delta}-2\sqrt{109}+1}{6}$	$\frac{\sqrt{\Delta}-4\sqrt{109}-1}{6}$	$\frac{\sqrt{\Delta}+3\sqrt{109}}{511\sqrt{109}+5335}$
$\frac{\sqrt{\Delta}-3\sqrt{109}}{511\sqrt{109}+5335}$	$\frac{\sqrt{\Delta}+3\sqrt{109}}{\sqrt{109}+13}$	$\frac{\sqrt{\Delta}-3\sqrt{109}}{\sqrt{109}+13}$	$\frac{\sqrt{\Delta}+3\sqrt{109}}{218\sqrt{109}+2276}$
$\frac{\sqrt{\Delta}-3\sqrt{109}}{218\sqrt{109}+2276}$	$\frac{\sqrt{\Delta}+4\sqrt{109}+1}{3\sqrt{109}+33}$	$\frac{\sqrt{\Delta}-4\sqrt{109}-1}{3\sqrt{109}+33}$	$\frac{\sqrt{\Delta}+4\sqrt{109}+1}{-3\sqrt{109}+33}$
$\frac{\sqrt{\Delta}-2\sqrt{109}+1}{-3\sqrt{109}+33}$	$\frac{\sqrt{\Delta}+2\sqrt{109}-1}{-3\sqrt{109}+33}$	$\frac{\sqrt{\Delta}-4\sqrt{109}-1}{-3\sqrt{109}+33}$	$\frac{\sqrt{\Delta}+21\sqrt{109}}{143\sqrt{109}+1493}$
$\frac{\sqrt{\Delta}-21\sqrt{109}}{143\sqrt{109}+1493}$			

6. Appendix

Below we give our implementation of Algorithm 4.1 in Sage. The user can input relatively prime square-free integers `d1` $= d_1 > 1$ and `d2` $= d_2 \geq 1$ and compute the CM points for the biquadratic extension $\mathbb{Q}(\sqrt{d_1}, \sqrt{-d_2})$. Alternatively, the user can input `d1` $= D$, `A` $= A < 0$, and `B` $= B$ as in (6) - (8) and compute the CM points for the cyclic extension $\mathbb{Q}\left(\sqrt{A\left(D + B\sqrt{D}\right)}\right)$. The CM type is the same as in Tables 1 and 2. If the program succeeds in finding a complete set of CM points $z_{\mathfrak{a}}$, the output is a list of ordered pairs $[a, b]$ such that $a, b \in E$ and

$$z = \frac{\sqrt{\Delta} - b}{a}$$

gives the decomposition

$$\mathfrak{a} = \mathcal{O}_F + \mathcal{O}_F z,$$

where $\mathfrak{a} \subset E$ is the fractional ideal generated by 1 and z. One such z is given for each ideal class in E. Here $\Delta = -d_2$ in the biquadratic case, and $\Delta = A(D + B\sqrt{D})$ in the cyclic case. The variable `f` appearing in the output is $\sqrt{d_1}$.

```
#Robert Cass, July 6, 2014, Computation of CM points for quartic abelian fields
M = QQ^4
d1 = 5
#F is the number field Q(f) obtained by adjoining a positive square root of d1
F.<f> = QuadraticField(d1,embedding=1)

if order(F.narrow_class_group()) == 1:

    G = F.galois_group()
    R.<t> = F[]

    #Use this code for the biquadratic case
    d2 = 3
    D = -d2

    #Use this code for the cyclic case
    #B = 2
    #A = -1
    #D = A*(d1+B*f)

    print "f = sqrt(", d1, "), F=Q(", f, "), E = F(", D, ")"

    #K is the number field Q(c) obtained by adjoining a square root of D to F, to is
    #the map from the relative number field2=F(e) to to K
    E.<e> = F.extension(t^2-D)
    K.<c> = E.absolute_field()
    fr, to = K.structure()
    Disc = F.discriminant()

    #Completes step 1
    CL = K.class_group()

    #Completes step 2
    OEBasis = E.integral_basis()

    #Completes step 3
    X = []
    Y = []
```

```
    for i in range(0,len(OEBasis)):
        Y.append((E(OEBasis[i])-(E.complex_conjugation()(OEBasis[i])))/(2*e))
        X.append((E.complex_conjugation()(OEBasis[i])+E(OEBasis[i]))/2)

    #This is the bound on ideal norms as in step 4 of the algorithm
    Bound = abs(D)*abs(Disc)*6/pi^2

    #Completes steps 4-5 and imposes the condition on CM type
    A = F.ideals_of_bdd_norm(Bound)
    C = []
    for d in range(1,len(A)):
        if len(A[d]) > 0:
            for J in A[d]:
                b=J.gens_reduced()[0]
                if G[0](b) > 0 and G[1](b) > 0:
                    C.append(b)
                if G[0](-b)> 0 and G[1](-b) > 0:
                    C.append(-b)
                if G[0](b*F.units()[0]) > 0 and G[1](b*F.units()[0]) > 0:
                    C.append(b*F.units()[0])
                if G[0](-b*F.units()[0]) > 0 and G[1](-b*F.units()[0]) > 0:
                    C.append(-b*F.units()[0])

    IList = []
    ZList = []

    if Mod(d1,4) == 1:
        s = to((1+f)/2)
    else:
        s = to(f)

    done = false

    for a in C:

    #Completes step 6
        L = []
        for b in (F.ideal(a).residues()):
            if F.ideal(a).divides(F.ideal(b^2-D)):
                L.append([a,b])

    #Completes step 7
        LL = []
        for pair in L:
            for i in range(0,len(OEBasis)):
                if (Y[i]*pair[0]).is_integral() and (X[i]+Y[i]*pair[1]).is_integral() and (X[i]-Y[i]*pair[1]).is_integral() and ((1/pair[0])*Y[i]*(D-pair[1]^2)).is_integral() and pair not in LL:
                    LL.append(pair)

    #Completes step 8 and explicitly verifies that the CM points give the appropriate decomposition of fractional ideals
        for n in range(0,len(LL)):
            N = to(e-LL[n][1])
```

```
            De = to(LL[n][0])
            if CL(K.ideal(N/De,1)) not in IList:
                a_1 = K.vector_space()[2](N/De)
                a_2 = K.vector_space()[2](s*N/De)
                a_3 = K.vector_space()[2](1)
                a_4 = K.vector_space()[2](s)
                V = M.span([a_1,a_2,a_3,a_4],ZZ)
                U = K.ideal(N/De,1).free_module()
                if U == V:
                    IList.append(CL(K.ideal(N/De,1)))
                    ZList.append([LL[n][0],LL[n][1]])
        if CL.order() == len(IList):
            done = true
            break
    if done:
        print "Found complete set of CM Points:"
        print ZList
    else:
        print "Failure to find complete set of CM points."
else:
    print "F does not have narrow class number 1."
```

References

[BSM] A. Barquero-Sanchez and R. Masri. *The Chowla-Selberg formula for abelian CM fields and Faltings heights*. 2014, preprint.

[BY] J. Bruinier and T. Yang. *CM-values of Hilbert modular functions*. Invent. Math. **163** (2006) 229-288.

[CSa] S. Chowla and A. Selberg. *On Epstein's zeta-function*. J. Reine. Angew. Math. **227** (1967) 86-110.

[CSb] S. Chowla and A. Selberg. *On Epstein's zeta-function I*. Proc. Nat. Acad. Sci. U. S. A. **35** (1949) 371-374.

[Den] C. Deninger. *On the analogue of the formula of Chowla and Selberg for real quadratic fields*. J. Reine. Angew. Math. **351** (1984) 171-191.

[IR] K. Ireland and M. I. Rosen. *A classical introduction to modern number theory*. Graduate Texts in Mathematics **84**. Springer-Verlag, New York, 1990.

[KKS] K. Kato, N. Kurokawa, and T. Saito. *Number theory 2. Introduction to class field theory*. Translated from the Japanese by M. Kuwata and K. Nomizu. Translations of Mathematical Monographs **240**. Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2011.

[Lan] S. Lang. *Algebraic number theory*. Second edition. Graduate Texts in Mathematics **110**. Springer-Verlag, New York, 1994.

[S+13] W. A. Stein et al. *Sage Mathematics Software (Version 5.13)*. The Sage Development Team, 2013. `http://www.sagemath.org`.

[Str10] M. Streng. *Complex multiplication of abelian surfaces*. 2010. PhD Thesis, Universiteit Leiden, Netherlands.

[SW] B. K. Spearman and K. S. Williams. *The conductor of a cyclic quartic field using Gauss sums*. Czechoslovak Math. J. **47** (1997) 453-462.

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