Moments of L-functions associated to Newforms of Squarefree Level

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What is a Modular Form?

Let

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \ \& \ ad - bc = 1 \right\}.$$

For $z \in \mathbb{H}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ let $gz = \frac{az+b}{cz+d}$. If $f : \mathbb{H} \to \mathbb{C}$ is holomorphic, satisfies

$$(cz + d)^{\kappa} f(gz) = f(z)$$

and is holomorphic at infinity, then f is a modular form of weight κ.

Level of a Modular Form

Let

$$\Gamma_0(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{q} \right\}.$$

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Modular Forms as a vector space

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Trace Formula

Theorem (Petersson Formula)

Let $\mathcal{B}$ be an orthogonal basis for $\mathcal{S}_\kappa(q)$, and define $\Delta_N(m,n) = c_\kappa \sum_{f\in\mathcal{B}} \frac{\lambda_f(m)\lambda_f(n)}{\langle f,f\rangle}$ *then*

$$\Delta_N(m,n) = \delta(m=n) + 2\pi i^{-\kappa} \sum_{\substack{c>0 \\ c\equiv 0(q)}} \frac{S(m,n;c)}{c} J_{\kappa-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

where $\langle\cdot,\cdot\rangle$ denotes the Petersson inner product, $S(x,y;c)$ denotes the Kloosterman sum, and $J_{\kappa-1}(x)$ denotes the J-Bessel function of order $k-1$, and $c_\kappa = \frac{\Gamma(k-1)}{(4\pi)^{k-1}}$.

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Unfortunately, newforms do not generally form a basis for the space of modular forms of fixed weight and level

Newform Trace Formula

Theorem (Newform Petersson Formula, Petrow and Young)

With $\Delta_q^*(m,n) := c_k \sum_{f \in \mathcal{H}_\kappa^*(q)} \frac{\lambda_f(m)\lambda_f(n)}{\langle f,f\rangle}$ *as before denote the RHS of the Petersson formula, then for squarefree* q *and even integer* κ*,*

$$\Delta_q^*(m,n) = \sum_{LM=q} \frac{\mu(L)}{\nu(L)} \sum_{\ell \mid L^\infty} \frac{\ell}{\nu(\ell)^2} \sum_{d_1,d_2 \mid \ell} c_\ell(d_1)c_\ell(d_2)$$

$$\sum_{u \mid (m,L) v \mid (n,L)} \frac{uv}{(u,v)} \frac{\mu(\frac{uv}{(u,v)^2})}{\nu(\frac{uv}{(u,v)^2})} \sum_{\substack{a \mid (\frac{m}{u}, \frac{uv}{(u,v)}) \\ b \mid (\frac{m}{u}, \frac{uv}{(u,v)})}} \sum_{\substack{e_1 \mid (d_1, \frac{m}{a^2(u,v)}) \\ e_2 \mid (d_2, \frac{n}{b^2(u,v)})}} \Delta_M(m,n)$$

where $c_\ell(d)$ *is jointly multiplicative and* $c_{p^n}(p^j) = c_{j,n}$ *with* $c_{j,n}$ *such that*

$$x^n = \sum_{j=0}^{n} c_{j,n} U_j\left(\frac{x}{2}\right),$$

where $U_n(x)$ *denotes the* n^{th} *Chebyshev Polynomial of the second kind.*

Approximate Newform Trace Formula

Lemma (Approximate Version of the Newform Trace Formula)

Let $m = \prod_p p^{m_i}$ *and* $n = \prod_p p^{m_i}$,

$$\Delta_q^*(m,n) = A_q(n,m) + O_{\kappa,\epsilon}(q^{-1+\epsilon}(mn)^{\frac{1}{4}+\epsilon})$$

where

$$A_q(n,m) = \begin{cases} \frac{\phi(q)}{q} \prod_{p|q} \sum_{\substack{n \leq n_i \\ m \leq m_i}} p^{-\frac{m_i+n_i}{2}} \prod_{p \nmid q} \delta(m_i = n_i) & mn \text{ is square} \\ 0 & \text{otherwise} \end{cases}$$

L-Functions associated to Newforms

Recall f has a fourier expansion at infinity,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{\kappa-1}{2}} e^{2\pi i n z}.$$

Further the $\lambda_f(n)$ are multiplicative.

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Further the $\lambda_f(n)$ are multiplicative. We associate to f an L-function $L(s, f)$ defined in the right half plane by the Dirichlet series:

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right),$$

where $\chi_0(p) = 0$ if $p | q$ and 1 if $p \not| q$.

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To this end, define the t^{th} shifted moment in level aspect to be:

$$\mathcal{M}^{(t)}(q,\kappa)_{\alpha_1,\alpha_2,\ldots,\alpha_t} := \sum_{f\in\mathcal{H}_\kappa^*(q)} \omega_f \prod_{i=1}^{t} L(\frac{1}{2}+\alpha_i, f),$$

where $\omega_f := \frac{c_\kappa}{\langle f,f\rangle}$ and the α_i satisfy $|Re(\alpha_i)| < \frac{1}{2}$ and $Im(\alpha_i) \ll q^\epsilon$ for any $\epsilon > 0$.

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where $\omega_f := \frac{c_\kappa}{\langle f,f\rangle}$ and the α_i satisfy $|Re(\alpha_i)| < \frac{1}{2}$ and $Im(\alpha_i) \ll q^\epsilon$ for any $\epsilon > 0$.

The main results are the explicit computation of asymptotics of the first two moments.

First Moment

Theorem (First Moment)

Let α *satisfy* $|Re(\alpha)| < \frac{1}{2}$ *and for all* $\epsilon > 0$, $Im(\alpha) \ll q^{\epsilon}$, *then*

$$\mathcal{M}_{\alpha}^{(1)}(q,k) = \frac{\phi(q)}{q} \prod_{p|q} \left(\frac{1}{(1 - p^{-(2+\alpha)})} \right) + O_k(q^{-1-\min(0,Re(\alpha))+\epsilon}).$$

where the implied constant depends on k *and* ϵ.

Second Moment

Theorem (Second Moment)

With ω_f as before and α, β shifts with real part less than $1/2$ in absolute value, and imaginary part bounded be q^ϵ for all $\epsilon > 0$ then for any $\epsilon > 0$, we have

$$\mathcal{M}^{(2)}_{\alpha,\beta}(q,k) = \frac{\phi(q)}{q}\Bigg(\zeta(1+\alpha+\beta)A_{\alpha,\beta}(q) + \left(\frac{2\pi}{\sqrt{q}}\right)^{2(\alpha+\beta)} \frac{\Gamma(\alpha+\frac{k}{2})\Gamma(\beta+\frac{k}{2})}{\Gamma(-\alpha+\frac{k}{2})\Gamma(-\beta+\frac{k}{2})}\zeta(1-\alpha-\beta)A_{-\alpha,-\beta}(q) + O(q^{-\frac{1}{2}-\min(Re(\alpha),Re(\beta))+\epsilon})$$

where the implied constant depends on κ and ϵ and $A_{(\alpha,\beta)}(q)$ is an explicit product over primes dividing q that for α and β with small real part is bounded and depends on q.

An Application of the First Two Moments

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Corollary (Nonvanishing at the Central Point)

Let $N_\kappa(q)$ denote the the set of all L-functions associated to newforms of weight κ and level q such that $L(\frac{1}{2}, f) > 0$, then $|N_\kappa(q)| \gg \frac{q}{\log(q)^2}$

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We will need that the weights are relatively uniform, precisely, it is know that $q\omega_f \ll \log(q)$

Proof of Corollary

Via the Cauchy-Schwarz inequality,

$$\Big|q \sum_{f\in\mathcal{H}_\kappa^*(q)} \omega_f L(\frac{1}{2}, f)\Big|^2 \leq \Big|\sum_{f\in N_\kappa(q)} q\omega_f\Big|\Big|q \sum_{f\in\mathcal{H}_\kappa^*(q)} \omega_f L(\frac{1}{2}, f)^2\Big|.$$

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But from the asymptotics, we have $\mathcal{M}_{0,0}^{(2)} \asymp \log(q)$ and $\mathcal{M}_0^{(1)} \asymp 1$
Rearranging

$$\Big|\sum_{f\in N_k(q)} q\omega_f\Big| \gg \frac{q}{\log(q)},$$

from which the claimed estimate follows

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- Bound the error term, in this case using Poisson summation

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