Bounding Components of Real Zero Sets of Bivariate Pentanomials

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Preliminaries

Definition

f is an n-variate $(n+k)$-nomial if $f \in \mathbb{C}[x_1, ...x_n]$ is of the form $f = \sum_{i=1}^{n+k} c_i x^{a_i}$ for $c_i \neq 0$. We call $A = \{a_1, ..., a_{n+k}\} \subset \mathbb{Z}^n$ the support of f.

Definition

The $\mathcal{A}$-discriminant variety of an n-variate $(n+k)$-nomial with support $\mathcal{A} = \{a_1, ..., a_{n+k}\} \subset \mathbb{Z}^n$ is defined as the closure of:

$$\nabla_A = (c_1, ...c_{n+k}) \in P_{\mathbb{C}}^{n+k-1} :$$

$$\exists \zeta \in (\mathbb{C}^n)^* \text{ with } f(\zeta) = 0, \frac{\partial f}{\partial x_i}(\zeta) = 0 \text{ for all } i \in \{1, ..., n\}$$

Parametrizing the $\mathcal{A}$-discriminant variety

- How do we efficiently compute the $\mathcal{A}$-discriminant variety?

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- For family $\mathcal{F}$ with support $A = \{a_1, ..., a_{n+k}\}$, we define an $(n+1) \times (n+k)$ matrix

$$\hat{A} = \begin{pmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{n+k} \end{pmatrix}$$

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- Define corresponding $(n+k) \times (k-1)$ matrix B whose columns form a basis of the right null of $\hat{A}$:

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_{n+k} \end{pmatrix}$$

Horn Kapranov Uniformization

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Theorem

For family $\mathcal{F}$ with $\hat{A}$ and B defined as on previous slide, ∇_A can be parametrized in $P(\mathbb{C})^{n+k-1}$ as the closure of the following:

$$\varphi(\nabla_A) = \{(b_1 \cdot \lambda)t^{a_1} : \cdots : (b_{n+k} \cdot \lambda)t^{a_{n+k}} \mid \lambda \in P(\mathbb{C})^{k-2}\}$$

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Also we can reduce φ to $\mathbb{R}^{k-1}$ as follows:

$$\overline{\varphi}(\nabla_A) = \{B^T log|B\lambda| \mid \lambda \in P(\mathbb{C})^{k-2}\}.$$

note: $\overline{\varphi}(\nabla_A)$ also induces a map from $\mathcal{F}$ into $\mathbb{R}^{k-1}$. Varying the coefficients of a polynomial varies the image of the polynomial under this map.

Definition

If $\overline{\varphi}(\nabla_{\mathcal{A}})$ denotes the reduced $\mathcal{A}$-discriminant variety, then a chamber of $\overline{\varphi}(\nabla_{\mathcal{A}})$ is a connected component of the complement of $\overline{\varphi}(\nabla_{\mathcal{A}})$ in R^{k-1}

Relevance of the $\mathcal{A}$-discriminant (part I)

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Fact

If $f, g \in \mathcal{F}$ correspond to points in the same chamber of $\overline{\varphi}(\nabla_{\mathcal{A}})$, then their real zero sets are isotopic.

Relevance of the $\mathcal{A}$-discriminant (part II)

Definition

In the setting of bivariate pentanomials, a chamber of $\overline{\varphi}(\nabla_{\mathcal{A}})$ is an outer chamber if its area is infinite and an inner chamber if its area is finite.

Relevance of the $\mathcal{A}$-discriminant (part II)

Definition

In the setting of bivariate pentanomials, a chamber of $\overline{\varphi}(\nabla_{\mathcal{A}})$ is an outer chamber if its area is infinite and an inner chamber if its area is finite.

Fact

The real zero sets of polynomials in the outer chambers can be completely characterized combinatorially using Viro's method.

Extended Example

$$\mathcal{F} = \{1 + x + y + ax^4y + bxy^4\}$$

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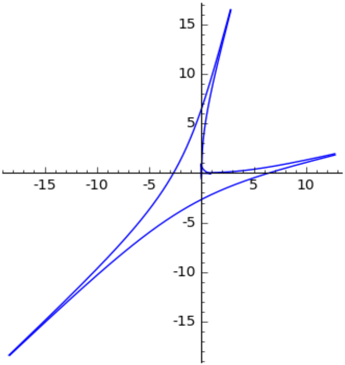

Figure: Reduced $\mathcal{A}$-discriminant amoeba for $\mathcal{F}$

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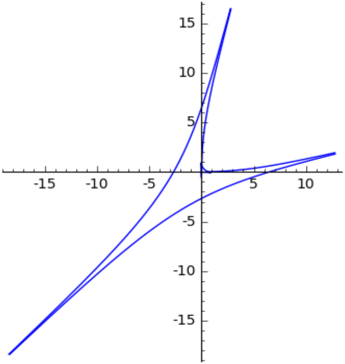

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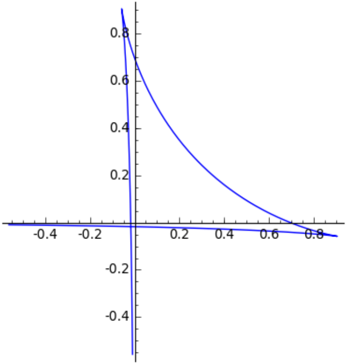

Figure: Quadrant 4 ($a, b < 0$) of unfolded $\mathcal{A}$-discriminant amoeba

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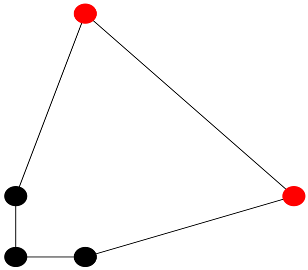

Figure: Signed Newton polygon for $\mathcal{F}_{(-,-)}$

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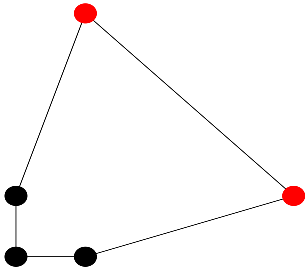

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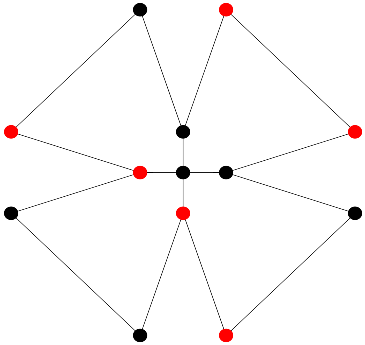
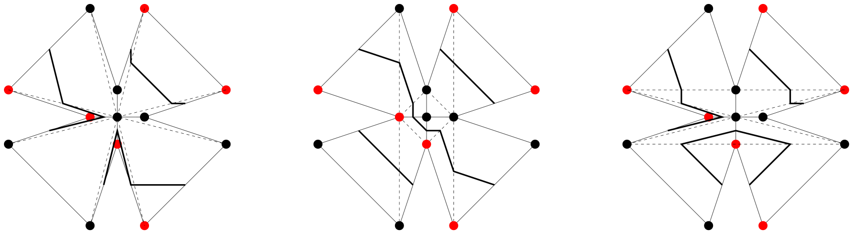


Figure: Expanded signed Newton polygon for $\mathcal{F}_{(-,-)}$



We can use Viro diagrams to completely categorize the topological types of polynomials in the outer chambers.

What about the inner chambers?

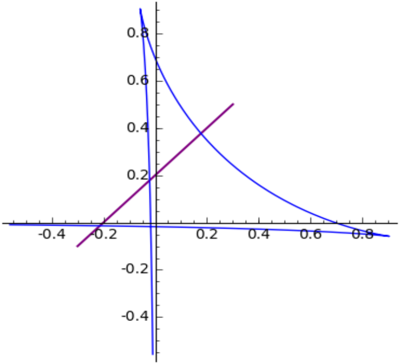


Figure: Quadrant 4 $(a, b < 0)$ of unfolded $\mathcal{A}$-discriminant amoeba

Main Theorem

Let $\mathcal{F}$ be a family of bivariate pentanomials of the following form (where $\alpha_i, \beta_i \in \mathbb{Z}$ are fixed):

$$\mathcal{F} = 1 + x + y + ax^{\alpha_1}y^{\alpha_2} + bx^{\beta_1}y^{\beta_2}$$

Theorem

Given $f, g \in \mathcal{F}$ lying in adjacent chambers of the reduced signed $\mathcal{A}$-discriminant amoeba of $\mathcal{F}$, $Non(f) = Non(g)$ and $|Comp(f) - Comp(g)| \leq 1$.

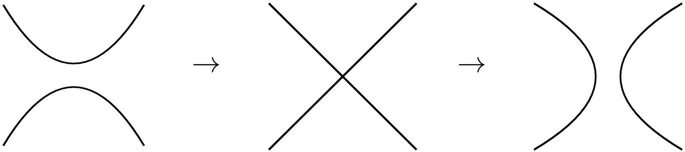
* Comp(f) (resp. Non(f)) denotes the number of compact (resp. non-compact) connected components in the real zero set of f

Changes in zero set crossing $\mathcal{A}$-discriminant

Local Extremum

$$\emptyset \quad \to \quad \bullet \quad \to \quad \bigcirc$$

Saddle



Outline of Proof

Consider family

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1. Given a polynomial f on the boundary between two chambers of the $\mathcal{A}$-discriminant for $\mathcal{F}$, there exists constant ϵ such that $f + \epsilon$ and $f - \epsilon$ lie in opposite chambers.

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2. We can ensure that f has exactly one degenerate root and that at that root, the surface defined by $f(x, y)$ attains either a local extremum or a saddle point.

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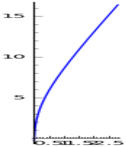
1. Given a polynomial f on the boundary between two chambers of the $\mathcal{A}$-discriminant for $\mathcal{F}$, there exists constant ϵ such that $f + \epsilon$ and $f - \epsilon$ lie in opposite chambers.
2. We can ensure that f has exactly one degenerate root and that at that root, the surface defined by $f(x, y)$ attains either a local extremum or a saddle point.
3. Assuming that ϵ is sufficiently small, the cross sections of the surface $f(x, y)$ at $f(x, y) = \pm\epsilon$ differ in number of (compact) connected components by at most 1.

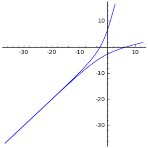
Another look at the signed reduced $\mathcal{A}$-discriminant

Signed reduced $\mathcal{A}$-discriminant for

$$\mathcal{F} = \{1 + x + y + ax^4y + bxy^4\}$$

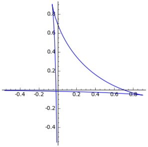
$\overline{\varphi}(-,+)$

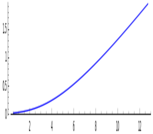




$\overline{\varphi}(+,+)$

$\overline{\varphi}(-,-)$





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- Undefined at $\lambda \in P^{k-1}$ for which $\lambda \cdot b_i = 0$ for some row b_i of the B matrix - here $\overline{\varphi}$ 'blows up' to infinity.

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- Undefined at $\lambda \in P^{k-1}$ for which $\lambda \cdot b_i = 0$ for some row b_i of the B matrix - here $\overline{\varphi}$ 'blows up' to infinity.
- For a bivariate pentanomial, we have (at most) 5 connected components partitioned between 4 quadrants.

Another look at the signed reduced $\mathcal{A}$-discriminant

Lemma

At most 3 connected components of the reduced signed $\mathcal{A}$-discriminant may lie in any given quadrant

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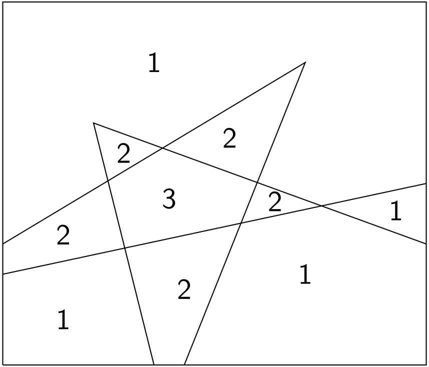
Lemma

The maximum 'depth' of a chamber in the signed reduced $\mathcal{A}$-discriminant is 3.

In the case with 3 components in a single quadrant, and 2 cusps (the maximum), the configuration of curves will look something like the following:



Bringing it All Together

Theorem

Given a bivariate polynomial f in a family of the form

$$\mathcal{F} = \{1 + x + y + ax^{\alpha_1}y^{\alpha_2} + bx^{\beta_1}y^{\beta_2} : a_i, b_i \in \mathbb{Z}_{\geq 0}\},$$

$$Comp(f) \leq 3$$

$$Tot(f) \leq 7.$$

Some Final Remarks

Remark

We are unsure whether the bound $Comp(f) \leq 3$ *is sharp - finding examples with multiple compact connected components would be a relevent pursuit.*

Remark

It is likely that with some working out of subtleties, our approach could give similar bounds for arbitary families of bivariate pentanomials.

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