# Higher-Dimensional Analogues of the Combinatorial Nullstellensatz

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# The Schwartz-Zippel Lemma

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How tight is the bound?

- roughly, tightest for polynomials of form $\sum_i \prod_j (x_i - s_j)$

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- note: we will assume K is a field throughout the talk
- enormous applications in many areas

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**Second Combinatorial Nullstellensatz (Alon 1999)**

Let $F \in K[x_1, \cdots, x_n]$, and suppose that $\deg(f) = \sum_{i=1}^{n} t_i$ for nonnegative integers $t_i$. Suppose further that the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in $F$ is nonzero. Then if $S_1, \cdots, S_n \in K$ with $\#S_i > t_i$ for each $i$, there is $s \in S_1 \times \cdots \times S_n$ such that

$$F(s) \neq 0.$$

# Two Applications

**Cauchy-Davenport Theorem (Cauchy 1813)**

If $p$ is a prime, and $A$, $B$ are two nonempty subsets of $\mathbb{Z}_p$, then

$$|A + B| \geq \min\{p, |A| + |B| + 1\}.$$

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**Theorem (Chevalley 1935, special case)**

Let $p$ be a prime, and let $P_1, \cdots, P_m \in \mathbb{Z}_p[x_1, \cdots, x_n]$. If $n > \sum_{i=1}^{m} \deg(P_i)$ and the polynomials $P_i$ have a common zero, then they have another common zero.

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- both classical results follow easily from Combinatorial Nullstellensatz

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- may have applications analogous to original applications of theorem
- $2 \times 2$ case already considered by Mojarrad et al.

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**Definition**

Let $\mathcal{G} = \{G_i : i \in \{1, \cdots, k\}\}$ be a set of polynomials. We say $\mathcal{G}$ is a $P$-family of polynomials if $G_i \in K[x_{n_{i-1}+1}, \cdots, x_{n_i}]$ for each $i$.

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**Definition (Cartesian Polynomial)**

Let $\mathcal{G} = \{G_i : i \in \{1, \cdots, k\}\}$ be a $P$-family of polynomials, and let $F \in K[x_1, \cdots, x_n]$. We say $F$ is $\mathcal{G}$-Cartesian if there are polynomials $H_1, \cdots, H_k \in K[x_1, \cdots, x_n]$ such that $\deg(H_i) \leq \deg(F) - \deg(G_i)$ for each $i$ and

$$F = \sum_{i=1}^{k} G_i H_i.$$

Further, if any such $P$-family of polynomials exists, we say $F$ is $P$-Cartesian.

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- the tuple (1,2,1,0,5) has $P$-reduction $(3, 6)$ for $P$ defined by $0 < 2 < 5$
- the polynomial $x_2x_3^4 + x_1x_2x_3^7x_4$ has $P$-support $\{(3, 4, 0), (2, 7, 1)\}$ for $P$ defined by $0 < 2 < 3 < 4$

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- in one dimension, $\#S = d(S)$

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- conclude $\sum f_1(s_1) \cdots f_k(s_k) F(s_1, \cdots, s_k) \neq 0$
- conclude $F$ does not vanish on all of $S_1 \times \cdots \times S_k$

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- analogue of Bézout's Theorem for many curves
- no direct analogue for three or more dimensions: consider many planes intersecting in a line

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- use lemma to find that $F$ vanishes on some $\prod Z(G_i)$
- use first generalized Combinatorial Nullstellensatz to show that $F$ is Cartesian

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- in three or more dimensions, infinite intersection no longer means intersection in a hyperplane
- much harder to find shared curve in three or more dimensions
- hence, difficult to show a polynomial is Cartesian from vanishing on a finite set

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To generalize the Schwartz-Zippel lemma to higher dimensions, starting with the $2 \times 2 \times \cdots \times 2$ case, by giving a bound on the intersection of a variety in $\mathbb{C}^{2k}$ with $S_1 \times \cdots \times S_k$, with the $S_i$ all 2-dimensional and finite.

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# Future Work

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- generalization would present improvements on Schwartz-Zippel in certain cases
- linked to generalized Combinatorial Nullstellensatz

Thank you!

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Thank you!