On Classification of the Unitarizability of Irreducible Representations of B_5

Étude Aro O'Neel-Judy

Northern Arizona University

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The Problem

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The Strategy

I needed to find a special basis in which all the matrices of this representation acquire a predetermined form.

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2. We successfully classified which representations of B_5 of dimension $d \leq 5$ are unitarizable!

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2. A qubit may be represented as a vector in a complex Hilbert space.
3. We can manipulate this quantum information by applying a unitary transformation (matrix).

What Words Mean

Definition (Braid Group)

The **braid group on** n**-strands** is given by

$$B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \forall\ i \in \{1, \ldots n-1\}$$
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Definition (Representation)

A **representation** of a group G is a pair (ρ, V), where V is a d dimensional vector space over $\mathbb{C}$ and ρ is a group homomorphism from G to the collection of $d \times d$ invertible matrices over $\mathbb{C}$.

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A representation ρ is **unitarizable** provided there exists a Hermitian inner product $\langle \cdot | \cdot \rangle_A$ such that $\langle \rho(g)v | \rho(g)w \rangle_A = \langle v | w \rangle_A$ for all $g \in G$ and for all $v, w \in V$.

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Note: The arbitrary inner product $\langle \cdot | \cdot \rangle_A$ may be related to the standard inner product via $\langle v|w \rangle_A = \langle Av|w \rangle$ for some matrix A.

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Under the standard inner product, a unitary matrix $\rho(g)$ has the property that $\langle \rho(g)v|\rho(g)v\rangle = \langle v|v\rangle$.

In other words, applying a unitary matrix to a vector **does not change the vector's length!**

Useful Tools

Definition (Adjoint)

Let A be a matrix, then we define the adjoint of $\rho(g)$ with respect to A via $\rho(g)^* = A^{-1}\rho(g)^\dagger A$, where $\dagger$ denotes complex conjugate transpose.

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Definition (Unitarizable Matrix)

A matrix $\rho(g)$ is unitarizable provided there exists a matrix A such that $\rho(g)\rho(g)^* = \rho(g)A^{-1}\rho(g)^\dagger A = I$.

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Classification

To classify the unitarizability of the representations of B_5, we need to check the unitarizability of $\tilde{\rho} = \chi(c) \otimes \rho(t)$ given $\rho(t)$.

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After further manipulation, we see that the above is equivalent to

$$0 = A\tilde{\rho}(\sigma_i) - ((\tilde{\rho}(\sigma_i))^\dagger)^{-1}A \tag{1}$$

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We see then that if $c\bar{c} = 1$, i.e. if c is on the unit circle, then $\dot{\rho}$ is unitarizable exactly when $\rho(t)$ is.

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We see then that if $c\bar{c} = 1$, i.e. if c is on the unit circle, then $\dot{\rho}$ is unitarizable exactly when $\rho(t)$ is.
An interesting question is whether there exists some c and some non-unitarizable representation $\rho(t)$ such that $\tilde{\rho}$ is unitarizable.

Results

1. Given $\rho(t)$, I set up some MatLab code which converts the equation matrix

$$0 = c\bar{c}(A\rho(t)(\sigma_i)) - ((\rho(t)(\sigma_i))^\dagger)^{-1}A$$

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2. I then solved the coefficient matrices for the Hecke $\rho(t) = H(t)$, and reduced-extended Burau $\rho(t) = \hat{\beta}(t)$ representations.
3. I found that for both H and $\hat{\beta}$ there was no c that satisfied the above equation for all σ_i.
4. **Collectively, Small Paul and I have fully classified which representations of B_5 of dimension $d \leq 5$ are unitarizable!**

Next Steps

1. Now that we are done with the representations of B_5, Paul and I have ambitions to classify representations of B_n for $n \neq 5$.

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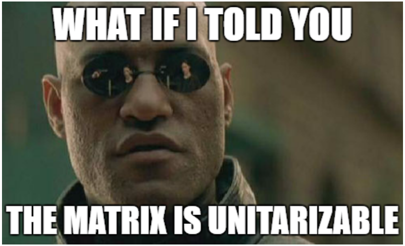
2. In this process, if we do not find any non-unitarizable representations $\rho(t)$ that can be unitarized with the right $\chi(c)$ then we will have shown by exhaustion that $\tilde{\rho}$ is unitarizable if and only if c is on the unit circle and $\rho(t)$ is unitarizable.

Thanks for listening!



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