Classification of Low Dimensional Unitarizable Representations of B_5

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1 Introduction

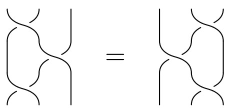
In this paper we classify the irreducible unitarizable representations of B_5 of dimension less than 5. This result expands on known results which have important applications in topological quantum computing and in representation theory. An application of unitary representations was given in [1] or see a survey paper by Rowell and Wang [2] for more information. Local unitary braid group representations can be used to construct a universal quantum computer on a collection of anyons. Such a computer would avoid decoherence and have a much lesser -if not completely eliminated-need for quantum error correcting codes.

Several important results already have been proven about the representations of the braid group. In general the representations of B_n have been classified for dimensions less than n, in paper a paper by Formanek [3] and then completed for dimension equal to n in [4]. Tuba and Wenzl have classified the representations of B_3 up to dimension five and give a condition which describes when the representation is unitarizable in [5] and [6]. We use the classifications and matrix forms of the representations of B_5 originally presented in [4] by E. Formanek, to classify which of the representations of B_5 are unitarizable.

2 Background

A representation ϕ is defined as a homomorphism from a group G into $GL_n(K)$, where K is some field and n is called the dimension of the representation. An irreducible representation ϕ is one such that the vector space that the representation acts on has no non-trivial invariant subspaces. More specifically, given a representation $\varphi : G \to GL_n(\mathbb{C})$ if there does not exist a $W \subset V$ such that for all $g \in G$, $\varphi(g)W \subset V$. In this paper we will be working with the representations of the braid group on five strands B_5 over the field of complex numbers which are at most five dimensional.

The braid group has several realizations; one is the motion of a collection of points in a disk over time, but it is perhaps easier to visualize as braids of strands of string where the operation is concatenation and topologically equivalent strings are identified. We give an example of a pair of equivalent braids of B_3 below which demonstrate the braid relation.



Symbolically the braid group on n-strands is given by the following:

$$B_n \doteq \langle \sigma_1, \sigma_2, \cdots \sigma_{n-1} \mid \sigma_{i-1}\sigma_i\sigma_{i-1} = \sigma_i\sigma_{i-1}\sigma_i \; and \; \sigma_i\sigma_j = \sigma_j\sigma_i \; if \; |i-j| \geq 2\rangle$$

The representation of the braid group is given by mapping each generator σ_i to some matrix. Together these matrices should satisfy the relationships which generate the braid group. Next we will give an introduction to some of the representations of B_n

A standard example of such a representation exists called the Burau representation. It is given by the map $\beta_n : B_n \to GL_n(\mathbb{C})$ where

$$\tilde{\beta_n}(\sigma_i) = \left[\begin{array}{c|cc|c} I_{i-1} & \multicolumn{2}{c|}{0} & 0 \\ \hline 0 & 1-t & t & 0 \\ & 1 & 0 & \\ \hline 0 & \multicolumn{2}{c|}{0} & I_{n-i-1} \end{array}\right]$$

The Burau representation is not irreducible; it always has a one dimensional invariant subspace given by $W = span(1, \ldots, 1)^T$. However the Burau representation is always the directed sum of the reduced Burau representation and W, the reduced Burau representation, denoted β, is given by the following mapping of the generators of B_n:

$$\beta_n(\sigma_1) = \left[\begin{array}{cc|c} -t & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & I_{n-3} \end{array}\right], \beta_n(\sigma_i) = \left[\begin{array}{c|ccc|c} I_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{n-i-2} \end{array}\right], \beta_n(\sigma_{n-1}) = \left[\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & t & -t \end{array}\right].$$

The reduced Burau representation is irreducible if and only if t is not a root of $f(z) = 1 + z + z^2 + \cdots + z^n$.

Formanek also uses a special form of the reduced Burau representation which is an extension of the reduced Burau representation on B_{n-1}. Let z be a root of $f(z) = 1 + z + z^2 + \cdots + z^n$, Then $\hat{\beta_n} : B_n \to GL_{n-2}(\mathbb{C})$ is the irreducible representation defined by: $\hat{\beta_n}(\sigma_i) = \beta_{n-2}(\sigma_i)$ for $1 \leq i \leq n-2$ and $\hat{\beta_n}(\sigma_{n-1}) = I - PQ$, where P is a column vector of size $n-2$ of all zeros except the last entry which is z and

$$Q = ((-1)^{n-2}z)(1, -(z+1), (1+z+z^2), \ldots, (-1)^{n-2}(1+z+z^2+\ldots+z^{n-2}).$$

A full motivation of this definition and proof that this representation is irreducible can be found in a paper by Formanek [3]. Our paper will work with the matrix form directly and will omit these details.

Another common representation is the standard representation introduced by Inna Sysoeva in [7] where she also proved that all n-dimensional irreducible representations of B_n for $n \geq 9$ are tensors of the standard representation with one dimensional representation $\chi(z)$. Define the one-dimensional representation $\chi(z) : B_5 \to \mathbb{C}$ by $\chi(z)(\sigma_i) = z$. Any one dimensional representation takes this form. The standard representation $s(y) : B_n \to (C)^n$ is given by:

$$s(t)(\sigma_i) = \begin{bmatrix} I_{i-1} & & & \\ & 0 & t & \\ & 1 & 0 & \\ & & & I_{n-i-1} \end{bmatrix}$$

The last representation used in the classification of all representations of B_5 was developed using Hecke algebras by V.F.R. Jones in [8]. We will denote this representation $\mu(t) : B_5 \to GL_5(\mathbb{C})$, where $t \in \mathbb{C}^*$. This paper will not detail the process of how the representations were constructed using Hecke algebras and will work only directly with the matrices, but in essence $\mu(t)$ is a representation of B_6 which is restricted to the first four generators of B_6 to give a representation of B_5. The representation μ is irreducible if t is not a root of $f(t) = (1 + t + t^2) * (t^2 + 1)$. If t is a root of $f(t) = t^2 + 1$ then $\mu(t)$ is the tensor product of $\chi(1)$, where $\chi(z)$ is defined by $\chi(z)(\sigma_i) = z$, and an irreducible representation $\hat{\mu}(t) : B_5 \to GL_4(\mathbb{C})$. In the case where t is a root of $f(t) = (1 + t + t^2)$ $\mu(t)$ is the tensor product of a one dimension representation and a representation which is equivalent to the reduced Burau representation. The source of the exact calculations is in [4].

Now we have all of the representations necessary to classify the representations of B_5 up to dimension five, using the results of the papers [3] and [4]. There are no two dimensional representations of B_5. All representations of B_5 which are three dimensional are of the Burau type.

A representation $\varphi : B_n \to GL_r(\mathbb{C})$ is of Burau type if $r \geq 2$ and it is equivalent to $\chi(y) \otimes \beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$ or $\chi(y) \otimes \hat{\beta}_n(z) : B_n \to GL_{n-2}(\mathbb{C})$. In dimension four, any irreducible representation φ is of Burau Type or φ is equivalent to $\chi(y) \otimes \hat{\mu}(z) : B_5 \to GL_4(\mathbb{C})$. Similarly in five dimensions the representation is equivalent to $\chi(y) \otimes \mu(z) : B_5 \to GL_4(\mathbb{C})$ where $\mu(z)$ is the irreducible Hecke representation described above. This paper focuses on classifying which of the above representations of B_5 are unitarizable, so it will now present some background on unitarizable representations.

A representation (φ, V) is said to be unitarizable if the vector space V can be equipped with a Hermitian inner product such that for all $g \in G$, $\langle \varphi(g)v | \varphi(g)w \rangle = \langle v|w \rangle$. Let $\langle v|w \rangle$ be the standard inner product on $\mathbb{C}^n$. Define $\langle v|w \rangle_A = \langle Av|w \rangle$ on $\mathbb{C}^n$. The following two lemmas are well-known results in linear algebra.

Lemma 2.1: Define the adjoint operator * with respect to $\langle \cdot | \cdot \rangle_A$ as $U^* = A^{-1}U'A$ where $'$ is the conjugate transpose. Then we have that $\langle Uv|Uw\rangle_A = \langle v|w\rangle_A$ for all $u, v \in \mathbb{C}^n$ if and only if $UU^* = I$.

Lemma 2.2: Define $\langle v|w\rangle_A = \langle Av|w\rangle$ for some positive Hermitian definite matrix A. Let $\langle v|w\rangle_*$ be some Hermitian inner product on $\mathbb{C}^n$ Then there exists some A such that $\langle v|w\rangle_* = \langle v|w\rangle_A = \langle Av|w\rangle$. This matrix A has values $a_{ij} = \langle e_i|e_j\rangle_1$ where e_i and e_j are elements of the standard basis of $\mathbb{C}^n$.

3 Main Results

First this paper presents a lemma which gives a helpful condition for a representation to be unitarizable.

Lemma 3.1: Let (φ, V) be a representation over a complex vector space. Then assume that there is a φ_X such that $\varphi_X(b) = X^{-1}\varphi(b)X$. If φ_X is unitary with respect to some Hermitian inner product $\langle u|v\rangle_*$ then φ is unitarizable.

Proof: By Lemma 2.2, we know that there is an A such that $\langle u|v\rangle_* = \langle u|v\rangle_A$, consider the adjoint operator with respect to this inner product *. By Lemma 2.1 we know that $\varphi(b)\varphi(b)^* = I$ for all $b \in G$. Along with reordering this gives $A(\varphi(b)')A^{-1} = \varphi_x(b)$ Next consider $((X')^{-1}A)^{-1}(\varphi(b)')((X')^{-1}A) = A^{-1}X'(X')^{-1}\varphi_x(b)'X'(X')^{-1}A = A^{-1}(\varphi(b)')A = \varphi_x(b)^{-1}$. So $X((X')^{-1}A)^{-1}(\varphi(b)')((X')^{-1}A)X^{-1} = X\varphi_x(b)^{-1}X^{-1} = \varphi(b)^{-1}$. Which tells us that $((X')^{-1}AX^{-1})^{-1}\ \varphi(b)'((X')^{-1}AX^{-1})\varphi(b) = I$. Thus for the adjoint * defined with respect to $\langle v|w\rangle_{(X')^{-1}AX^{-1}}$, we have $\varphi(b)^*\varphi(b) = I$. By Lemma 2.1 we know that φ is unitary with respect to $\langle v|w\rangle_{(X')^{-1}AX^{-1}}$. Thus φ is unitarizable as required.

Theorem 3.1 The reduced Burau representation is unitarizable when $J_{n-1} = XX^\dagger$ for some X.

Proof: This follows from Lemma 3.1 and the following construction from Squier in [9]. Define P_{n-1} and J_{n-1} as below:

$$P_{n-1} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & s & & \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & s^{n-1} \end{bmatrix}, J_{n-1} = \begin{bmatrix} s+s^{-1} & -1 & \dots & 0 \\ -1 & s+s^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 0 & \dots & -1 & s+s^{-1} \end{bmatrix}$$

By Theorem 2.3 in [1] we have that $\beta(z)_S = P_{n-1}^{-1}\beta(z)P_{n-1}$ is unitary with respect to J_{n-1}, this implies by Lemma 3.1 that the reduced Burau representation is unitary if the J_{n_1} matrix is positive definite. In other words that $J_{n_1} = XX^\dagger$ for some X.

Theorem 3.2: The standard representation $s(t) : B_5 \to \mathbb{C}^5$ of B_5 is unitary if and only if t is on the unit circle.

Proof: If t is on the unit circle then $S_iS_i^\dagger = I$ so the matrices mapped to by the generators are unitary with respect to the standard inner product. Now if we assume that t is not on the unit circle we can prove that the standard representation is not unitary by using the following system of equations procedure.

Using Lemma 2.2 we have $UU^* = I$ which is equivalent to $U^\dagger A - AU^{-1} = 0$. Then with an appropriate symbolic equation solver we can define an a matrix of variables A and solve a system of equations for them. The exact MatLab code appears in the appendix, we provide some pseudo code below. Here $\phi(\sigma_i) = B_i$.

```
A = symbolicMatrix(n)
for i = 1:4
E_i = ConjugateTranspose(B_i)*A - A*inverse(B_i) == 0
V_i = eqnToMatrix(E_i)
end
V = [ V1; V2; V3; V4 ]
solve(V)   %%note we do this by hand as auto systems failed
```

Using this and some rather tedious calculations we have proven the following theorems.

Theorem 3.4: The Hecke representations $\mu(t) : B_5 \to \mathbb{C}^5$ where t is not a zero of $f(t) = (t^2+1)(t^2+t+1)$ and the specialization of the reduced Burau representation $\hat{\beta} : B_5 \to \mathbb{C}^3$ is never unitarizable.

Proof: My research partner Etude O'Neel-Judy has included a proof of this in his paper.

Theorem 3.5: The specialization of the Hecke representation $\hat{\mu} : B_5 \to \mathbb{C}^4$ is never unitarizable.

Proof: Both of these theorems are proved using calculations on the systems of equations given by the pseudo-code above. First note that if t is a root of $t^2 + t + 1$ then $\hat{\mu}$ is equivalent to the reduced Burau representation as was shown in [3]. It remains to check $t = \pm i$ we will show the case $t = i$.

Upon calculation row 28 and 31 of the system give $a_{2,3} = a_{3,4}$ and $a_{1,3} = a_{2,5}$ respectively. Row 2 gives $-a_{1,1} - a_{1,3} - a_{2,4} = 0$, and row 8 gives $-a_{1,1} + ia_{1,3} - a_{2,4} = 0$, which implies $(1+i)a_{1,3} = 0$ so $a_{1,3} = 0$, $a_{1,1} = 0$, and $a_{2,4} = 0$.Then row 12 gives $a_{3,3} = ia_{1,4}$ and row 15 gives $a_{3,3} = a_{1,4}$ so $a_{1,4} = 0$ and $a_{3,3} = 0$. Row 23 is $a_{1,2} + a_{1,3} + a_{2,4} = 0$ so $a_{1,2} = 0$. Row 17 gives $-ia_{1,5} - a_{2,3} = 0$ and Row 20 gives $-a_{1,5} - a_{3,4} = 0$ so adding rows gives $(1 - i)a_{1,5} = 0$. Thus A has an all zero row and is not invertible, which is a contradiction.

At this point in the paper we have shown which of the basic representations of B_5 are unitarizable. Thus to finish the classification we need to examine the way the tensor product affects the unitarizablity of the representation.

Theorem 3.5: Given a representation φ of B_n and the one dimensional representation $\chi(z)$ of B_n, the representation $\varphi \otimes \chi(z)$ is unitarizable if and only if there exists a Hermitian inner product $\langle \cdot | \cdot \rangle_A$ such that for all σ_i, $\langle \varphi(\sigma_i)u | \varphi(\sigma_i)v \rangle_A = c\langle u|v\rangle$ where c is some positive real with $|z| = \frac{1}{\sqrt{c}}$.

Proof: If there exists an A such that $\langle \varphi(g)v | \varphi(g)w \rangle_A = c\langle v|w\rangle_A$ for all $v, w \in \mathbb{C}^n$ for some positive real c, then pick your favorite z such that $|z| = \frac{1}{\sqrt{c}}$. Now $\langle \chi \otimes \varphi(g)v | \chi \otimes \varphi(g)w \rangle_A = \langle z * \varphi(g)v | z * \varphi(g)w \rangle_A = |z|^2(c\langle v|w\rangle_A) = \langle v|w\rangle_A$. So assume that $\chi(z) \otimes \varphi$ is unitarizable. On the other hand by similar computation $c = \frac{1}{|z|^2}$.

Main Theorem: The following list gives every irreducible unitary representation of B_5 up to dimension five.

$d = 1$: $\chi(z)$ where $|z| = 1$.

$d = 2$: No unitary irreducible representations.

$d = 3$: No unitary irreducible representations.

$d = 4$: The Burau type representation $\chi(z) \otimes \beta(t)$ when $|z| = 1$ and the previously described J_n matrix is positive definite.

$d = 5$: The standard representation $\chi(z) \otimes s(t) : B_5 \to \mathbb{C}^5$ when $|t| = 1$ and $|z| = 1$.

Proof: As we have previously established conditions where each irreducible representation is unitary before tensor product we need to check the tensor product's effect. We do this using a system of equations derived from the unitary condition and the previous theorem. Namely we build a system as before but with $|z|^2 U^\dagger A^{-1} - AU^{-1} = 0$.

This paper presents the calculations for the tensor product with the standard, Burau, and specialized Hecke $\hat{\mu}$. My research partner Etude ONeel-Judy, presents them for the Hecke μ and specialized Burau $\hat{\beta}$ in his paper. Note that in the following calculation the row refers to the row of the matrix given by our matlab code and we set $c = |z|^2$. Also note that if $c = 1$ then our condition reduces to the case where the representation is unitary so in the following calculations we assume $c \neq 1$.

For the Burau representation $\beta(t)$: equation 17 gives $(c-1)a_{1,1} = 0$, equation 19 gives $(c-1)a_{1,3} = 0$, equation 20 gives $(c-1)a_{1,4} = 0$, equation 34 gives $(c-1)a_{1,2} = 0$, equation 37 gives $(c-1)a_{1,5} = 0$. Then if $c \neq 0$ the whole first row is zero which contradicts that A is invertible.

For the standard representation $s(t)$: equation 51 gives $(c-1)a_{1,1} = 0$, equation 52 gives $(c-1)a_{1,2} = 0$, equation 78 gives $(c-1)a_{1,3} = 0$, equation 29 gives $(c-1)a_{1,4} = 0$, equation 30 gives $(c-1)a_{1,5} = 0$. Then if $c \neq 0$ the whole first row is zero which contradicts that A is invertible. So we only have the unitary case.

For the specialized Hecke representation, note that we just present the case $t = i$ ($t = -i$ is similar). Equation 34 gives that $(c-1)a_{1,1} = 0$, equation 36 gives $(c-1)a_{1,3} = 0$, equation 31 gives $(c-1)a_{5,5}$, equation 33 gives $(c-1)a_{5,3}$. We split cases $c = i$ or $c \neq i$. Then if $c = i$, equation 5 implies $a_{1,4} = 0$, equation 19 implies that $a_{1,2} = 0$, equation 6 implies that $a_{1,5} = 0$. So this implies that A is not invertible, a contradiction. If $c \neq i$, then equation 35 implies that $a_{1,2} = 0$, equation 37 implies that $a_{1,4} = 0$, and equation 22 implies $a_{1,5} = 0$. So again A would not be invertible, a contradiction. Thus this reduces to the case where specialized Hecke representation is unitarizable.

4 Conclusion

Previous results in the literature classify irreducible representations of B_n up to dimension n, we have extended these to a partial result classifying the unitary representations of B_n up to dimension n. The classification of the unitary conditions of the Burau representation had already been performed. In addition to this, the standard representation has clear unitarizablity conditions. We proved conditions on the remaining irreducible representations of B_5 of low dimension and none are unitary. This is an important step of the classification as for B_n with $n \geq 9$ there are only the Burau and standard. Thus the work presented in this paper leaves only the classification of the unitary conditions of Hecke representations for B_n in $n = 6, 7, 8$ and the unitary representations of B_4. It is likely that the required classification of B_4 would likely follow from mapping known representations of B_3 into B_4.

References

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Appendix

Here is the exact code used to get the system of equations for the entries of A. It is altered from code provided by my research partner Etude Judy-ONeel.

```
function M = hecke_matrices(H_1,H_2,H_3,H_4)
[n,~] = size(H_1);
syms t
syms c
X = sym('x', n);

%Need empty vectors to fill with entries of Yi
V1 = [];
V2 = [];
V3 = [];
V4 = [];
%variable vector
```

```
x = [];
%X has to satisfy the following equation matrices
%note to test without the tensor product of the constant factor remove
    c
Y1 = X*inv(H_1) - c*H_1'*X == 0;
Y2 = X*inv(H_2) - c*H_2'*X == 0;
Y3 = X*inv(H_3) - c*H_3'*X == 0;
Y4 = X*inv(H_4) - c*H_4'*X == 0;
%This puts the above equation matrices in vector form
for i= 1:n
    for j = 1:n
        V1 = [V1 Y1(i,j)];
        V2 = [V2 Y2(i,j)];
        V3 = [V3 Y3(i,j)];
        V4 = [V4 Y4(i,j)];
        x = [x X(i,j)];
    end
end
%master equation vector
V = [V1 V2 V3 V4];
%convert all equations from equation matrices Yi into a single
    coefficient
%matrix
M = equationsToMatrix(V,x);
```