Probability of Easily approximating the Positive Real Roots of Trinomials

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Joint work with Laurel Newman

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Overview

1. Motivation/Background
2. Using $\mathcal{A}$-Discriminants to Find Number of Positive Real Roots
3. Estimations Using Algebraic Geometry
4. How Signs of Coefficients Effect Estimation
5. Failure Sets and Regions
6. Update on Experiments and Effect of Exponents on Probability

Motivation/Background

Polynomial System Solving

$$f(f_1, f_2) = \begin{cases} c_1 x^{a_1} y^{b_1} + c_2 x^{a_2} y^{b_2} + c_3 x^{a_3} y^{b_3} \\ c_4 x^{a_4} y^{b_4} + c_5 x^{a_5} y^{b_5} + c_6 x^{a_6} y^{b_6} \end{cases}$$

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Standard Quadratic: $f(x) = c_1 + c_2 x + c_3 x^2$

Quadratic Formula:

$$\frac{-c_2 \pm \sqrt{c_2^2 - 4c_1 c_3}}{2c_1}$$

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$$c_2^2 - 4c_1c_3 < 0 : 0\ \mathbb{R}^+ \text{ Real Roots}$$

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Logarithmic Version:

$$\log|c_2^2| = \log|4c_1c_3|$$

$$2\log|c_2| = \log|4| + \log|c_1| + \log|c_3|$$

$$\text{-}\log|c_1| + 2\log|c_2| - \log|c_3| = \log|4|$$

Definitions for $\mathcal{A}$-Discriminant Formula

- Support$(f(x)) = \begin{bmatrix} a_1 & a_2 & \dots & a_{n+2} \end{bmatrix}$
  - $n \in \mathbb{Z}^+$
  - $f = c_1 x^{a_1} + c_2 x^{a_2} + \dots + c_{n+2} x^{a_{n+2}}$

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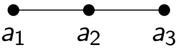
- Honest: $DimNewt(f) = n$ where n is the number of variables
  - f(x)=$c_1 x^{a_1} + c_2 x^{a_2} + c_3 x^{a_3}$
  - Support$(f(x)) = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$



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- Also let b := any generator for the right $\mathbb{Z}$-Nullspace of $\widehat{A}$
- Then,

  $b_1 \log|c_1| + \cdots + b_{n+2} \log|c_{n+2}| \overset{\geq}{\underset{<}{=}} b_1 \log|b_1| + \cdots + b_{n+2} \log|b_{n+2}|$

  $\Rightarrow\ b \cdot \log|c| \overset{\geq}{\underset{<}{=}} b \cdot log|b|$

Discriminant Example

Discriminant Inequalities

$$b_1 \log|c_1| + \cdots + b_{n+2} \log|c_{n+2}| \substack{>\\=\\<} b_1 \log|b_1| + \cdots + b_{n+2} \log|b_{n+2}|$$

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Quadratic Case

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$$c_1 + c_2 x + c_3 x^2$$

1. $\widehat{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, b= $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

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Quadratic Case

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1. $\widehat{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, b= $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$
2. $-1 \log|c_1| + 2 \log|c_2| - 1 \log|c_3| \underset{<}{\overset{>}{=}} -1 \log|-1| + 2 \log|2| - 1 \log|-1|$

   $\Rightarrow -\log|c_1| + 2\log|c_2| - \log|c_3| \underset{<}{\overset{>}{=}} \log|4|$

   $\Rightarrow (-1, 2, -1) \cdot \log|c| \underset{<}{\overset{>}{=}} \log|4|$

Algebraic Geometry Estimation Tools

Archimedean Newton Polytope

$$Archnewt(f) := conv\{(a_j, -\log|c_j|) | j \in \{1, ..., t\}\}$$

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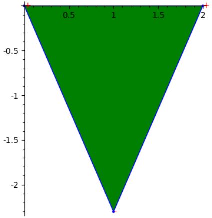
Examples:

$$1 - 10x + x^2$$
$$(0, -\log|1|), (2, -\log|1|)$$
$$(1, -\log|-10|)$$



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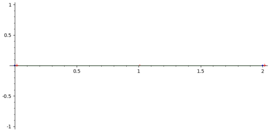
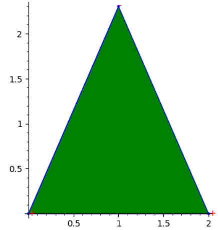


$$1 - \frac{1}{10}x + x^2$$
$$(0, -\log|1|), (2, -\log|1|)$$
$$(1, -\log|-\frac{1}{10}|)$$



Algebraic Geometry Estimation Tools (cont)

Positive Archimedean Tropical Variety

When $f(x) = \Sigma_{j=1}^{t} c_j x^{a_j}$,

$$\text{Trop}_+(f) := \left\{ w \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \max_{j \in \{1,\ldots,t\}} \left| c_j e^{a_j w} \right| \text{ is attained at indices } j, j' \\ \text{with } c_j c_{j'} < 0. \end{array} \right\}$$

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Example: $f(x) = 1 - 10x + x^2$

$$|1e^{0w}| = |-10e^{1w}| > |1e^{2w}| \Rightarrow$$

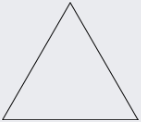
$$\ln(1) = \ln(10e^{w}) > \ln(e^{2w}) \Rightarrow$$

$$0 = \ln 10 + w > 2w \Rightarrow$$

$$-\ln(10) = w > 2w$$

Estimates of Real Roots based on Archnewt

Estimates for Standard Quadratic

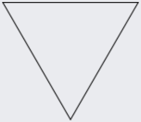


$(-1, 2, -1) \cdot \log|c| < 0$

0 $\mathbb{R}^+$ Roots

$(-1, 2, -1) \cdot \log|c| = 0$

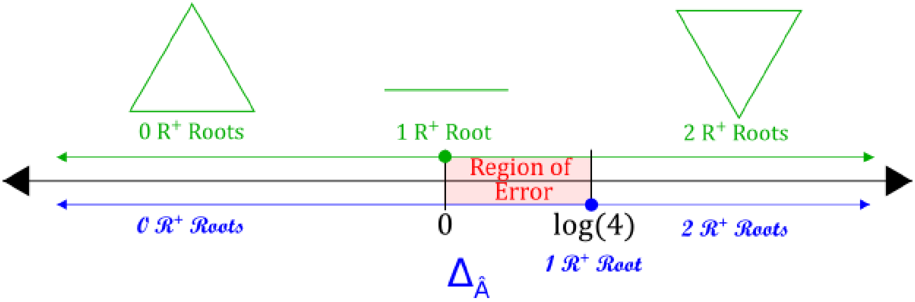
1 $\mathbb{R}^+$ Root



$(-1, 2, -1) \cdot \log|c| > 0$

2 $\mathbb{R}^+$ Roots

Failure Region for Standard Quadratic



General Failure Sets and Region

Failure Region

$$f(x) = c_1 + c_2 x^{a_1} + c_3 x^{a_2}$$

Error Region: $0 < b \cdot \log|c| < b \cdot \log|b|$

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Failure Set

Set of coefficients $c = (c_1, c_2, c_3)$ such that $sign(c) = \pm\, sign(b)$ and b $\cdot \log|c|$ lies in the failure region

Signs

Recall

$$f(x) = c_1 x^{a_1} + c_2 x^{a_2} + c_3 x^{a_3}$$

b := any generator for the right $\mathbb{Z}$-Nullspace of $\widehat{A}$

Signs and Failure Possibility

- If $sign(b_i)$ and $sign(c_i)$ are equal or opposite for all i $\in \{1, 2, 3\}$, there is a possibility that $Trop^+$ will not accurately estimate the positive zero set
- In all other cases, topology of $Trop^+$ is constant with respect to the coefficients

Update on Experiments and Why Exponents Matter

Probability Various Trinomials Lie in Error Region

- $f(x) = c_1 + c_2x + c_3x^2$
  - 5.9895%
- $f(x) = c_1 + c_2x^{26} + c_3x^{50}$
  - 5.9744%
- $f(x) = c_1 + c_2x^{99} + c_3x^{100}$
  - 0.4471%
- $f(x) = c_1 + c_2x^{19} + c_3x^{20}$
  - 1.6041%

Thank you for listening!
Special thanks to Professor Rojas, Joann Coronado, and the National Science Foundation!

$Trop^{+}(f)$ as an Estimate for Positive Roots

Theorem

If $T = \dim(\text{Newt}(f)) + 1$, then any point in $\log|Z_{+}(f)|$ is within $\log(t-1)$ of some point in $\text{Trop}_{+}(f)$, and furthermore $\text{Trop}_{+}(f)$ is isotopic to $\log|Z_{+}(f)|$.

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