Probability of Easily Approximating Positive Reals Roots of Trinomials

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Outline

1. Notation
2. Failure Probability vs. Exponent Ratio
3. Failure Probability vs. Variance Ratio
4. Upper Bounding Failure Probability vs. Variance Ratio
   - Small sigma: linear
   - Large sigma: x^{-k}

Notation

Univariate Trinomials

Let $f(x) = c_1 x^{\alpha_0} + c_2 x^{\alpha_1} + c_3 x^{\alpha_2}$

- $\alpha_0 < \alpha_1 < \alpha_2$
- $c_i \sim N(0, \sigma_i)$
- generally, $\alpha_0 = 0$

Notation

Spread

$$\text{spread}(f) := \frac{\min(\alpha_1 - \alpha_0, \alpha_2 - \alpha_1)}{\alpha_2 - \alpha_0}$$

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- $\text{spread}(c_1 x^{\alpha_0} + c_2 x^{\frac{\alpha_0+\alpha_2}{2}} + c_3 x^{\alpha_2}) = 0.5$
- as $\alpha_1 \to \alpha_0$ or α_2, $\text{spread}(f) \to 0$

Trinomial Exponent Ratio

Experimental Consideration

What is the relationship between the spread of a trinomial f and its failure probability?

Trinomial Exponent Ratio

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What is the relationship between the spread of a trinomial f and its failure probability?

Method:

- fix α_2
- iterate α_1 from $[1, \alpha_2 - 1]$
- 1,000,000 trials per ratio
- generate new random standard Gaussian coefficients each trial

Trinomial Exponent Ratio: Results I

$f = c_1 + c_2 x^{\alpha_1} + c_3 x^{100}$

- 99 exponent ratios
- scipy's `curve_fit` function

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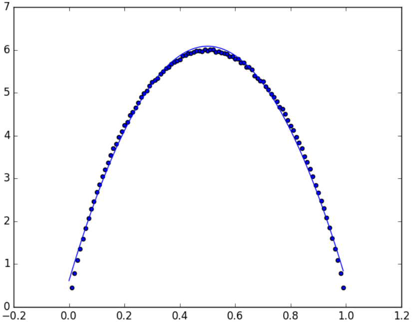

Figure: $\frac{\alpha_1}{100}$ vs. Failure Probability

$$h(x) = 0.61353465 + 21.87751589x - 21.86653471x^2$$

Trinomial Exponent Ratio: Results II

$f = c_1 + c_2 x^{\alpha_1} + c_3 x^{100}$

- 99 exponent ratios
- $h(x) = 0.61353465 + 21.87751589x - 21.86653471x^2$

$f = c_1 + c_2 x^{\alpha_1} + c_3 x^{25}$

- 24 exponent ratios
- $h(x) = 0.70218905 + 21.39398914x - 21.38648046x^2$

$f = c_1 + c_2 x^{\alpha_1} + c_3 x^{1987}$

- $\alpha_1 \in [19, 1900]$
- $h(x) = 0.65875168 + 21.56950267x - 21.5027753x^2$

Trinomial Exponent Ratio: Results III

$f = c_1 x^{24} + c_2 x^{a_1} + c_3 x^{626}$

- 100 exponent ratios
- x-axis $\frac{24}{\alpha_1}$

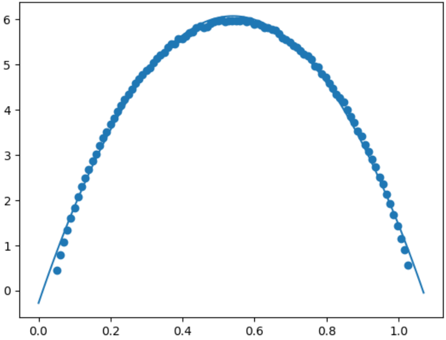

Figure: $\frac{24}{\alpha_1}$ vs. Failure Probability

$$h(x) = -0.27225719 + 23.51542209x - 21.77854389x^2$$

Trinomial Exponent Ratio: Conjectures

Experimental Hypotheses

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- The graph of the failure probability as a function of trinomial spread is, roughly, a parabola or ellipse
- Failure probability appears to never exceed 6%
- Failure probability also depends on variance ratios

Quadratic Variance Ratio

Experimental Consideration

What is the relationship between the failure probability of f, a quadratic polynomial, and $\frac{\sigma_2}{\sigma_1}$, recalling that $c_i \sim N(0, \sigma_i)$?

Quadratic Variance Ratio

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Method:

- 100 values of σ_2 in $[0.1, 10]$
- 1,000,000 trials per ratio
- generate c_1 and c_3 from standard Gaussian distributions, and c_2 from $N(0, \sigma_2)$ each trial

Varying the standard deviation of c_2:

- $\sigma_2 \in [0.1, 10]$

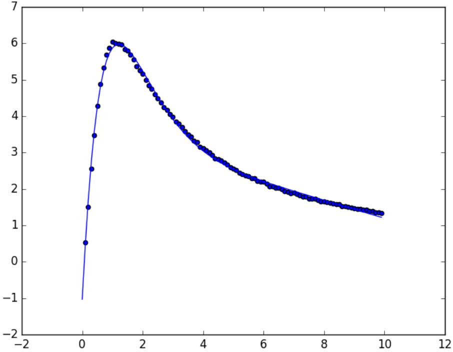

Figure: Quadratic σ_2 vs. Failure Probability

$$h(x) = -1.03061413 + 15.572038x^{1.0356945}e^{-1.04617418x} + 1.76374323xe^{-0.20716401x}$$

Quadratic Variance Ratio: Results II

Varying the standard deviation of c_3:

- $\sigma_3 \in [0.1, 100]$

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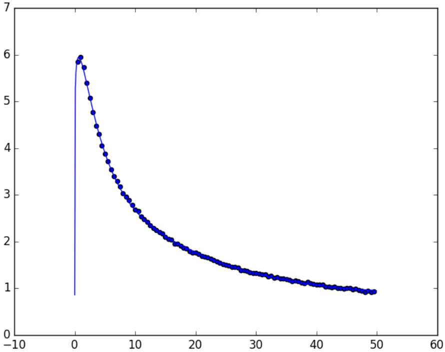

Figure: Quadratic σ_3 vs. Failure Probability

$$h(x) = 0.85961511 + 6.15174179x^{0.13562741}e^{-0.26987804x} + 0.35691471xe^{-0.10525011x}$$

Variance Ratio for Trinomials with Small Spread

Experimental Consideration

What is the relationship between the failure probability of $f = c_1 + c_2x^{99} + c_3x^{100}$ and $\frac{\sigma_2}{\sigma_1}$, recalling that $c_i \sim N(0, \sigma_i)$?

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Method:

- 100 values of σ_2 in $[0.1, 60]$
- 1,000,000 trials per ratio
- generate c_1 and c_3 from standard Gaussian distributions, and c_2 from $N(0, \sigma_2)$ each trial

Tight Trinomial Variance Ratio: Results I

Varying the standard deviation of c_2:

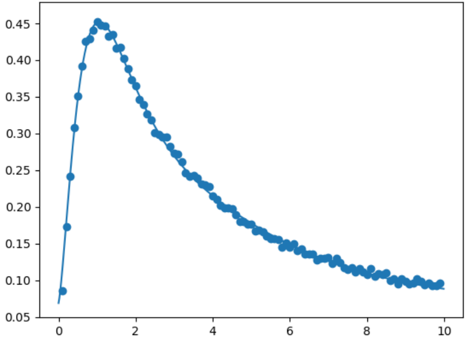

Figure: σ_2 vs. Failure Probability

$$h(x) = -0.06450709 + 0.18826155x^{0.55247034}e^{-0.15034146x} - 1.03096168xe^{-1.09906311x}$$

Tight Trinomial Variance Ratio: Results II

Varying the standard deviation of c_1:

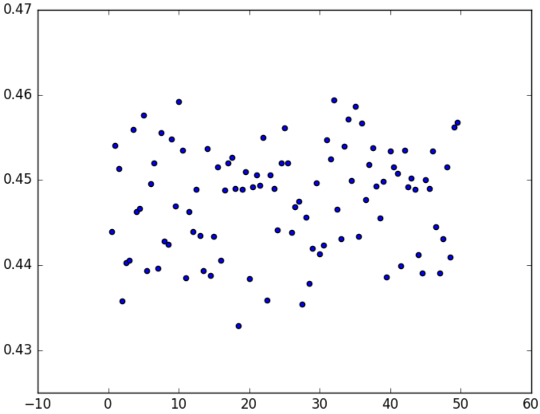

Figure: σ_1 vs. Failure Probability

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New Experimental Questions

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- Can we transform the fit functions into upper bounds?

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  Idea: Could using multiple simple piecewise functions approximate the failure probabilities?
- Can we extract meaning from the coefficients of the fit functions?
  Idea: Do the coefficients have a relationship to the exponent spread of the polynomial?
- Can we transform the fit functions into upper bounds?
  Idea: Can we find specific coefficients that upper bound the failure probabilities for all exponent spreads?

Can we simplify the fit functions in some way?

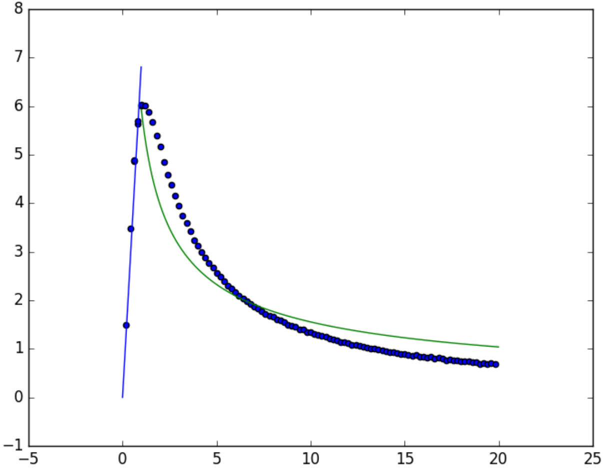

Figure: Piecewise linear and x^{-k} fit functions for failure probability vs. σ

Piecewise Variance Ratio: $\sigma_2 \leq 1$

Experimental Consideration

What is the minimum slope that upper bounds the failure probability when $\sigma_2 \leq 1$?

$f(x) = c_1 + c_2 x + c_3 x^2$

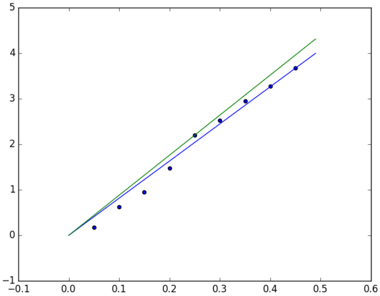

Figure: Linear upper bound and fit lines for failure probability vs. $\sigma \leq 1$

Piecewise Variance Ratio: $\sigma_2 \leq 1$

Experimental Consideration

What is the minimum slope that upper bounds the failure probability when $\sigma_2 \leq 1$, and what is its relationship to the trinomial's spread?

Piecewise Variance Ratio: $\sigma_2 \leq 1$

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What is the minimum slope that upper bounds the failure probability when $\sigma_2 \leq 1$, and what is its relationship to the trinomial's spread?

Method:

- 10 exponent ratios in $[0.1, 1]$
  - 10 values of σ_2 in $[0.1, 1]$
  - 100,000 trials per σ_2
  - generate c_1 and c_3 from standard Gaussian distributions, and c_2 from $N(0, \sigma_2)$ each trial
  - find upper bound curve of form $g(x) = ax$
- per trinomial exponent ratio, average 10 values of a

Piecewise Variance Ratio: $\sigma_2 \leq 1$ Results

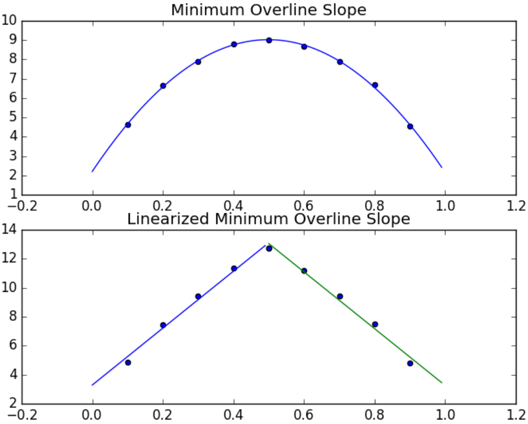

Figure: Minimum slopes for upper bound line vs. trinomial exponent ratio

Piecewise Variance Ratio: $\sigma_2 \leq 1$ Results

$$g(x) = a\sqrt{\frac{\max(\alpha_1, \alpha_2 - \alpha_1)}{\alpha_2}}x$$

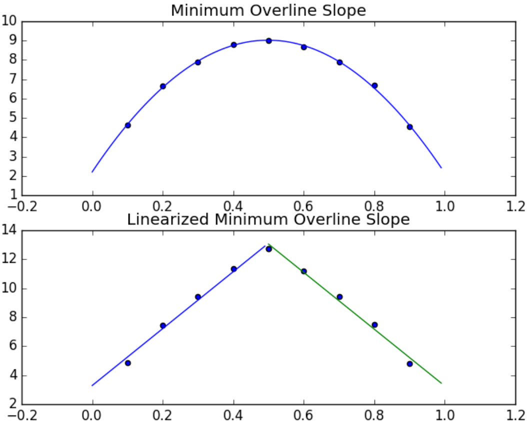

Figure: Minimum slopes for upper bound line vs. trinomial exponent ratio

Piecewise Variance Ratio: $\sigma_2 \geq 1$

Experimental Consideration

Finding a function of the form $g(x) = ax^{-k}$ which is an upper bound for failure probability when $\sigma_2 \geq 1$.

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Experimental Consideration

Finding a function of the form $g(x) = ax^{-k}$ which is an upper bound for failure probability when $\sigma_2 \geq 1$.

Method:

- 10 exponent ratios in $[0.1, 1]$
  - 10 values of σ_2 in $[1, 20]$
  - 1,000,000 trials per σ_2
  - generate c_1 and c_3 from standard Gaussian distributions, and c_2 from $N(0, \sigma_2)$ each trial
  - fit data to $g(x) = ax^{-k}$ using scipy's `curve_fit` function
  - increment k until g is an upper bound curve
- per exponent ratio, average 10 values of k

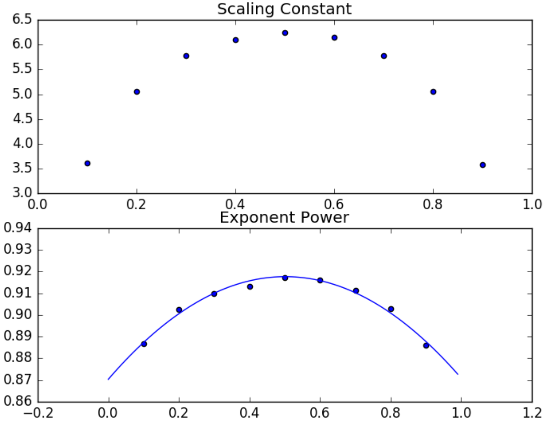

Figure: Upper bound constants and exponents vs. trinomial exponent ratios

Piecewise Variance Ratio: $\sigma_2 \geq 1$

Experimental Consideration

What is the minimum upper bound curve of the form $g(x) = ax^{-0.9}$ for failure probability when $\sigma_2 \geq 1$.

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Experimental Consideration

What is the minimum upper bound curve of the form $g(x) = ax^{-0.9}$ for failure probability when $\sigma_2 \geq 1$.

Method:

- 10 exponent ratios in $[0.1, 1]$
- 10 values of σ_2 in $[1, 20]$
- 1,000,000 trials per σ_2
- generate c_1 and c_3 from standard Gaussian distributions, and c_2 from $N(0, \sigma_2)$ each trial
- fit data to $g(x) = ax^{-0.9}$ using scipy's `curve_fit` function
- increment a until g is an upper bound curve
- select maximum a

$g(x) = 6.5x^{-0.9}$

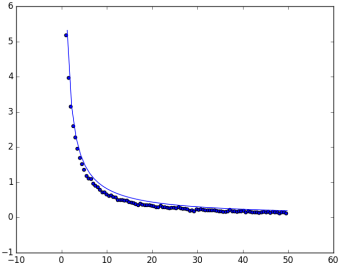

Figure: $f(x) = c_1 + c_2 x + c_3 x^2$

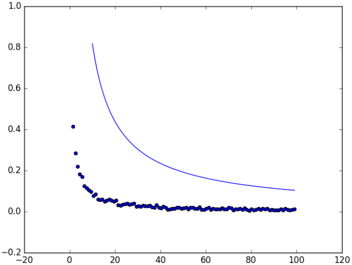

Figure: $f(x) = c_1 + c_2 x^{99} + c_3 x^{100}$

Further Work

- Tighter bound lines (especially for $\sigma \geq 1$)?
- Coefficient meaning for $\sigma \geq 1$?
  - Possible dependence on spread?
- Can we establish theoretical bounds that support these experimental results?
- Can we otherwise characterize the polynomials which fail?

Thank you for listening!

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