# Counting Points on Arbitrary Curves over Prime Power Rings

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# Overview

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*Suppose $k \in \mathbb{N}$, $f \in \mathbb{Z}[x]$ is not identically zero in $(\mathbb{Z}/\langle p \rangle)[x]$, and $\zeta_0 \in \mathbb{Z}/\langle p \rangle$ is a non-degenerate root of $\tilde{f} := f \bmod p$. Then there is a unique $\zeta \in \mathbb{Z}/\langle p^k \rangle$ with $\zeta_0 = \zeta \bmod p$, and $f(\zeta) = 0 \bmod p^k$.*

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## Proposition

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  - p = 3, n = 3, k = 4, j = 1
- f has $9 \cdot 3^{(3-1)(4-1)} = 6561$ roots over $(\mathbb{Z}/\langle 3^4 \rangle)^3$

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  - $\zeta = \mu + p^{i-1}\zeta_{i-1}$
  - $k_{i,\zeta} = k_{i-1,\mu} - s_{i-1}$
  - $f_{i,\zeta}(x) := \left[\frac{1}{p^{s_{i-1}}} f_{i-1,\mu}(\zeta_{i-1} + px)\right] \bmod p^{k_{i,\zeta}}$

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  - There is an edge from $(f_{j,\mu}, k_{j,\mu})$ to $(f_{i,\zeta}, k_{i,\zeta})$ if and only if $j = i - 1$, and there is degenerate root $\zeta_{i-1}$ of $\tilde{f}_{j,\mu}$ with $s(f_{j,\mu}, \zeta_{i-1}) \in \{2, ..., k_{i,\mu} - 1\}$, and $\zeta = \mu + p^{i-1}\zeta_{i-1}$

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# Total Count

$$N_{p,k}(f) = p^{(k-1)(n-1)} n_{p,k}(f) + \left( \sum_{\substack{\zeta_0 \in (\mathbb{F}_p)^n \\ s(f,\zeta_0) \geq k}} p^{n(k-1)} \right) + \left( \sum_{\substack{\zeta_0 \in (\mathbb{F}_p)^n \\ s(f,\zeta_0) \in \{2,\dots,k-1\}}} p^{n(s(f,\zeta_0)-1)} N_{p,k-s(f,\zeta_0)}(f_{1,\zeta_0}) \right)$$

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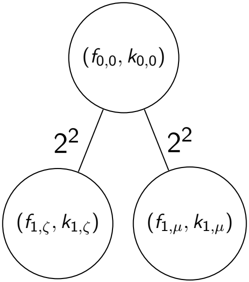

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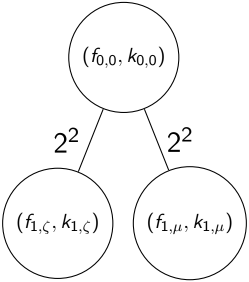
# Counting Example 1 (cont)

- Left node has 8 roots
- Right node has 8 roots
- Total count $= 64 + 2^2(8) + 2^2(8) = 128$ over $(\mathbb{Z}/\langle 2^4 \rangle)^2$



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Roots of f over $(\mathbb{Z}/\langle 2^2 \rangle)^2$: {(0,0),(0,2),(2,0),(2,2)}

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- For curves (n = 2) one can attain complexity $dkp^{1+o(1)}$ if one has access to algorithms which count over $\mathbb{F}_p$ in time $(log\ p)^{O(1)}$

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- Computing the intermediate $f_{i,\zeta}$ can be sped up with some interpolation tricks.
- In one variable, [BLQ13] showed that $O(dk\ log\ p)$ is possible. Two variable case is open!

# The End