Bounds for Coefficients of the $f(q)$ Mock Theta Function and Applications to Partition Ranks (Part 2)

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Partitions of even/odd rank

We utilize our effective bound on $\alpha(n)$ to resolve the following conjecture:

Conjecture (Hou and Jagadeesan [2], 2017)

If $r = 0$ (resp. $r = 1$), then we have that

$$N(r,2;a)N(r,2;b) > N(r,2;a+b)$$

for all $a, b \geq 11$ (resp 12).

Hou and Jagadeesan demonstrated a similar result for the modulo-three rank-counting functions $N(r,3;n)$ for $r = 0, 1, 2$, but their methods do not work modulo two.

Growth of $N(r, 2; n)$

Theorem (Gomez-Zhu)

For $n \geq 4$,

$$N(r, 2; n) = \frac{H(n)}{36 I(n)^2}\left(1 - \frac{1}{I(n)}\right) + (-1)^r R_2(n)$$

where $H(n) := \pi^2\sqrt{3}e^{I(n)}$ *and*

$$|R_2(n)| \leq (8.17 \times 10^{30})e^{I(n)/2}$$

Growth of $N(r,2;n)$

We will make use of an effective bound on the partition function due to Lehmer:

Theorem (Lehmer, 1938)

For all $n \geq 1$,

$$p(n) = \frac{2\sqrt{3}}{24n-1}\left(1 - \frac{1}{l(n)}\right)e^{l(n)} + E_p(n)$$

where $|E_p(n)| \leq (1313)e^{l(n)/2}$.

Sketch of Proof

We substitute the asymptotic formulas for $p(n)$ and $\alpha(n)$ into the relation

$$N(r,2;n) = \frac{p(n) + (-1)^r \alpha(n)}{2}$$

and then bound the resulting error

$$R_2(n) := (-1)^{n-1} \frac{\pi}{\sqrt{6} l(n)} e^{l(n)/2} + \frac{1}{2}(E_p(n) + E(n)). \quad \square$$

We will use the previous theorem to prove the following crucial inequalities:

Lemma (Gomez-Zhu)

For $r = 0$ (resp. $r = 1$), we have that

$$\frac{H(n)}{36l(n)^2}\left(1 - \frac{1}{l(n)}\right)^2 < N(r, 2; n) < \frac{H(n)}{36l(n)^2}\left(1 - \frac{1}{l(n)^2}\right)$$

for all $n \geq 16$ (resp. 15).

This lemma places $N(r, 2; n)$ into a "nice" window, one which we manipulate to resolve the conjecture.

By our previous theorem,

$$\frac{H(n)}{36I(n)^2}\left(1-\frac{1}{I(n)}\right)-|R_2(n)|<N(r,2;n)$$

and

$$N(r,2;n)<\frac{H(n)}{36I(n)^2}\left(1-\frac{1}{I(n)}\right)+|R_2(n)|.$$

Thus, we can bound $N(r,2;n)$ for large enough n

$$\frac{H(n)}{36I(n)^2}\left(1-\frac{1}{I(n)}\right)^2<N(r,2;n)<\frac{H(n)}{36I(n)^2}\left(1-\frac{1}{I(n)^2}\right)$$

so long as the coefficient of $e^{I(n)}$ bounding $|R_2(n)|$ is not too large.

Sketch of Proof

How large is too large? Given that $|R_2(n)| \leq (8.17 \times 10^{30})e^{l(n)}$, we need n large enough to satisfy

$$8.17 \times 10^{30} < \frac{\pi^2\sqrt{3}}{36l(n)^3}\left(1 - \frac{1}{l(n)}\right)e^{l(n)/2}.$$

Computation shows that $n > 4647$ will do, but we require our bounds to hold for significantly smaller n to resolve the conjecture.

We thus analyze the remaining $n < 4647$ using the Online Encyclopedia of Integer Sequences, which contains the values of $p(n)$ and $\alpha(n)$ for $1 \leq n \leq 10^4$, and find that $N(r, 2; n)$ falls into our window for $n \geq 15$ when $r = 0$ (resp. $n \geq 16$ when $r = 1$). $\square$

We now prove the complete conjecture. Assume $16 \leq a \leq b$ and let $b = Ca$ where $C \geq 1$. We have just demonstrated that

$$N(r,2;a)N(r,2;Ca) > \frac{H(a)H(Ca)}{1296 l(a)^2 l(Ca)^2}\left(1-\frac{1}{l(a)}\right)^2\left(1-\frac{1}{l(Ca)}\right)^2$$

and

$$N(r,2;a+Ca) < \frac{H(a+Ca)}{36 l(a+Ca)^2}\left(1-\frac{1}{l(a+Ca)^2}\right).$$

Thus, we need only find a such that our lower bound for $N(r,2;a)N(r,2;Ca)$ exceeds our upper bound for $N(r,2;a+Ca)$.

Proving the Conjecture

This is equivalent to finding a such that

$$e^{T_a(C)} > \frac{12\sqrt{3} I(a)^2 I(Ca)^2}{\pi^2 I(a+Ca)^2} S_a(C),$$

where

$$T_a(C) := I(a) + I(Ca) - I(a+Ca)$$

and

$$S_a(C) := \frac{\left(1 - \frac{1}{I(a+Ca)^2}\right)}{\left(1 - \frac{1}{I(a)}\right)^2 \left(1 - \frac{1}{I(Ca)}\right)^2}.$$

Or, taking logarithms of both sides,

$$T_a(C) > \log\left(\frac{12\sqrt{3} I(a)^2 I(Ca)^2}{\pi^2 I(a+Ca)^2}\right) + \log S_a(C).$$

Proving the Conjecture

We first observe that, as functions of C, T_a is strictly increasing and S_a is strictly decreasing, so we need only find a which satisfy our inequality for $C = 1$

$$T_a(1) > \log\left(\frac{12\sqrt{3}I(a)^2I(Ca)^2}{\pi^2I(a+Ca)^2}\right) + \log S_a(1).$$

We then make use of the fact that $I(Ca)^2/I(a+Ca)^2 \leq 1$ for all a since $I(a+Ca) > I(Ca)$ to reduce our inequality to

$$T_a(1) > \log\left(\frac{12\sqrt{3}I(a)^2}{\pi^2}\right) + \log S_a(1).$$

Proving the Conjecture

For which a is this final relation true? We calculate $T_a(1)$ and $S_a(1)$ and find that $a \geq 16$ suffice, and thus the conjecture is proven for such $a, b \geq 16$.

The remaining cases of $11 \leq a, b \leq 15$ (resp. $12 \leq a, b \leq 15$) for $r = 0$ (resp. $r = 1$) are then checked manually by comparing $N(r, 2; a)$, $N(r, 2; b)$, and $N(r, 2; a + b)$. □

Further Speculation

With this result, we might ask if we can obtain similar convexity results for other moduli? That is, do we have, for $t > 3$ and $0 \leq r < t$,

$$N(r, t; a)N(r, t; b) > N(r, t; a + b)$$

for all $a, b \geq C(t)$, where $C(t) > 0$ is an explicit constant depending only on the modulus t?

If we were able to find finite algebraic formulas describing $N(r, t; n)$ analogous to ours for larger t, this conjecture would be resolved as in the case of $t = 2$. However, no such formulas are yet known.

Thank you

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References

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