Algebraic signatures for a non-local obstruction and sunflowers

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Summary

- Review of Neural Codes and Neural Ideals
- Algebraic signature of non-local obstruction to non-closed convexity
- Proof of signature
- Sunflowers
- Part of the algebraic signature of sunflowers
- Closed realizations for sunflowers

Neural Ideal

Definition

A **neural code**, C, is a set of codewords, which are binary strings of length n. We also denote the codewords by the index of 1's in the string, e.g., 0110=23.

Definition

The **neural ideal of a code** is the ideal of the polynomial ring $\mathbb{F}_2[x_1, \ldots, x_n]$ that consists of all polynomials whose zeros are precisely the codewords in the code C.

$$J_C = < \chi_\nu \mid \nu \in \mathbb{F}_2^n \setminus C >,$$

$$\chi_\nu = \prod_{i|\nu_i=1} x_i \prod_{j|\nu_j=0} (1 + x_j)$$

Note: $\mathbb{F}_2^n = \{0, 1\}^n$

The canonical form of J_C

Definition

A **pseudo-monomial** is a polynomial with the form $\chi = \prod_{i\in\mu} x_i \prod_{j\in\tau}(1+x_j)$ for $\mu, \tau \subset \{1, \ldots, n\}$, where $\mu \cap \tau = \emptyset$.

A pseudo-monomial χ_{ν_1} is **minimal** in J_C if no other pseudo-monomial χ_{ν_2} in J_C divides χ_{ν_1}.

Definition

The **canonical form** of J_C is

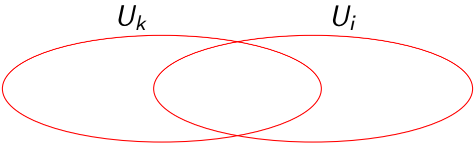
$$CF(J_C) = \{minimal\ pseudo\text{–}monomials\ of\ J_C\}$$

Fact: The canonical form generates the neural ideal.

Proof for algebraic signature of an obstruction to closed convexity

Definition

A **receptive field** $U_i \subset \mathbb{R}^d$ is the region of space that triggers the firing of the i-th place cell in a group of n cells, indexed using the set $[n] = \{1, \ldots, n\}$



Pseudo-monomials and receptive field relationships

Some pseudo-monomials and the corresponding receptive field relationships (Curto et al. [1])

$$x_{i_1}x_{i_2}x_{i_3} \implies U_{i_1} \cap U_{i_2} \cap U_{i_3} = \emptyset, U_{i_1} \cap U_{i_2} \neq \emptyset, \quad U_{i_2} \cap U_{i_3} \neq \emptyset, U_{i_1} \cap U_{i_3} \neq \emptyset$$

$$x_{i_1}x_{i_2}(x_{i_3}+1) \implies U_{i_1} \cap U_{i_2} \cap U_{i_3} \neq \emptyset$$

$$x_{i_1}(x_{i_2}+1)(x_{i_3}+1) \implies U_{i_1} \subset (U_{i_2} \cup U_{i_3})$$

Notation: $U_{i_1} \cap U_{i_2} \cap U_{i_3} = U_{i_1 i_2 i_3}$

Algebraic signature of an obstruction to closed convexity

Definition

An **algebraic signature** for some property is a subset of an algebraic set that encodes the property in question.

For example: $C15$ is a non-closed convex code (Goldrup and Phillipson [2])

$$
\begin{aligned}
CF(J_{C15}) = \{&(x_5+1)(x_2+1)x_1, (x_5+1)x_4x_1, (x_5+1)x_4(x_3+1),\\
&(x_3+1)x_2(x_1+1), x_4x_2x_1, x_4(x_3+1)x_2,\\
&x_5x_4x_2, x_4x_3x_1, x_5x_2(x_1+1), x_5(x_4+1)(x_1+1),\\
&x_3(x_2+1)x_1, (x_4+1)x_3(x_2+1), x_5x_3x_1,\\
&x_5x_3x_2, x_5(x_4+1)x_3\}
\end{aligned}
$$

$$
\begin{aligned}
AS(C15) = \{&x_1x_3(x_2+1), x_2x_5(x_1+1), x_3x_5(x_4+1), x_5x_3x_1,\\
&x_5x_3x_2, x_3(x_4+1)(x_2+1), x_2(x_1+1)(x_3+1)\}
\end{aligned}
$$

Theorem about algebraic signature of an obstruction to closed convexity

Theorem

Let C be a code on n neurons. Let $i, j, k, l, m \in [n]$. Suppose the canonical form of the neural ideal of C has the following subset of pseudo-monomials:

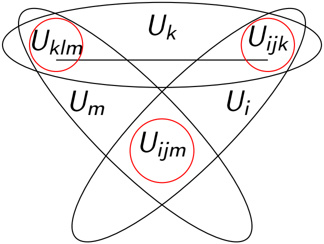
$$\{x_i x_k(x_j+1), x_j x_m(x_i+1), x_k x_m(x_l+1), x_m x_k x_i, x_m x_k x_j, x_k(x_l+1)(x_j+1), x_j(x_i+1)(x_k+1)\}$$

Then, the code C is non-closed convex, and we refer to the set of pseudo-monomials as the algebraic signature for this obstruction.

Proof for algebraic signature of an obstruction to closed convexity

Lemma (1)

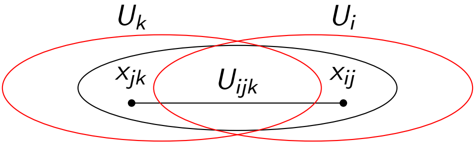
Let C be a convex neural code. If $x_ix_kx_m, x_ix_k(x_j+1), x_jx_m(x_i+1), x_kx_m(x_l+1) \in CF(J_C)$, then the sets U_{ijk}, U_{ijm}, and U_{klm} are nonempty and disjoint and the points $y_{ijk} \in U_{ijk}$, $y_{ijm} \in U_{ijm}$, and $y_{klm} \in U_{klm}$ are not colinear.



Proof for algebraic signature of an obstruction to closed convexity

Lemma (2)

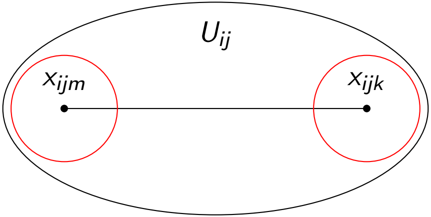
Let C be a neural code and U_i, U_j, and U_k be nonempty, convex sets in $\mathbb{R}^d$. If $x_j(x_i+1)(x_k+1) \in CF(J_C)$, then any line drawn between a point $x_{ij} \in U_{ij}$ and a distinct point $x_{ijk} \in U_{jk}$ passes through the nonempty intersection U_{ijk}.



Proof for algebraic signature of an obstruction to closed convexity

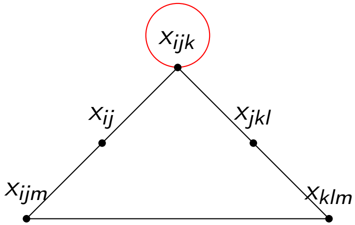
Lemma (3)

Let C be a neural code with convex receptive sets U_{ijk} and U_{ijm} in its realization. If $x_i x_k x_m \in CF(J_C)$, then any line that passes between a point in U_{ijk} and U_{ijm} must contain a point in $U_{ij} \setminus (U_{ijk} \cup U_{ijm})$.



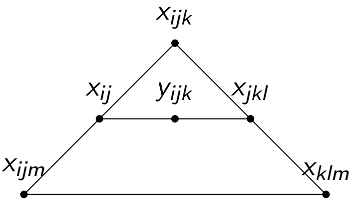
Proof for algebraic signature of an obstruction to closed convexity

Using Lemmas 1,2, and 3, we build the following triangle.



Proof for algebraic signature of an obstruction to closed convexity

$$(x_k + 1)x_j(x_i + 1) \implies U_j \subset (U_i \cup U_k)$$



Corollary to theorem

Corollary

If a code C satisfies the following

1. *The code contains the codewords ij, ijk, ijm, jkl, klm*
2. *No codewords contain ikm or jkm*
3. *Every codeword that contains k also contains j or l*
4. *No codewords that contains j also contains i or k*

then C is not closed covex.

Sunflower codes (Jeffs [3])

Definition (Sunflower code)

Let $n \geq 2$. Define the sunflower code, $S_n \subset 2^{[2n+2]}$ ($[2n+2] = \{1, \ldots, 2n+2\}$), to be the combinatorial code that consists of the following codewords:

1. $\emptyset$,
2. All codewords of the form $\sigma(n+1)$ for σ a nonempty proper subset of $[n]$,
3. $n+1+j$ for $1 \leq j \leq n+1$,
4. $(1 \cdots (i-1)(i+1) \cdots n)(n+1)(n+1+i)$ for $1 \leq i \leq n$,
5. the codeword $1 \cdots n(n+1)(2n+2)$, and
6. the codeword $(n+2)(n+3) \cdots (2n+2)$.

$$S_3 = \{\emptyset, 5, 6, 7, 8, 14, 24, 124, 34, 134, 234, 2345, 1247, 1346, 12348, 5678\}$$

Sunflower codes (Jeffs [3])

Theorem

Let $\sigma \in C$. Let $i, j, k \in \sigma$. Then,

$$U_i \cap U_j = U_j \cap U_k = U_i \cap U_k = U_{ijk} \neq \emptyset$$

$$\Updownarrow$$

$$x_i x_j (x_k + 1), x_i (x_j + 1) x_k, (x_i + 1) x_j x_k \in CF(J_C)$$

Corollary (Part of AS(S_n))

For $i, j, k \in \{n+2, n+3, \ldots, 2n+2\} \in S_n$

$$x_i x_j (x_k + 1), x_i (x_j + 1) x_k, (x_i + 1) x_j x_k \in CF(J_{S_n})$$

$$S_3 = \{\emptyset, 5, 6, 7, 8, 14, 24, 124, 34, 134, 234, \\ 2345, 1247, 1346, 12348, 5678\}$$

Sunflower codes (Jeffs [3])

Conjecture

The algebraic signature for the sunflower code S_n must have the following properties.

1. $\{x_i x_j (x_k + 1), x_i (x_j + 1) x_k, (x_i + 1) x_j x_k\} \subset AS(S_n)$ *for* $i, j, k \in \{n+2, n+3, \ldots, 2n+2\}$
2. $x_i (x_{n+1} + 1) \in AS(S_n)$ *for* $i \in [n]$ *and* $x_{n+1} \prod_{j \in [n]} (x_j + 1) \in AS(S_n)$.
3. $x_1 \cdots x_{n+1} (x_{2n+2}) \in AS(S_n)$
4. $x_i x_j (x_k + 1)$ *for* $i \in \{n+2, \ldots, 2n+2\}$, $j, k \in ([n] \setminus \{i\} \cup \{n+1\} \cup \{i\})$
5. $x_i \prod_{j \in \tau} (x_j + 1) \notin AS(S_n)$ *for* $i \in \{n+2, \ldots, 2n+2\}$ *and* $\tau \subset [n+1]$
6. $(x_{2n+2} + 1) x_n \cdots x_1 \in AS(S_n)$

Sunflower codes (Jeffs [3])

Theorem (Closed convexity of sunflowers)

Although the sunflower codes are not open convex, they are closed convex. The sunflower code S_2 is closed convex in $\mathbb{R}^2$. The sunflower code S_n, $n \geq 3$, is closed convex in $\mathbb{R}^3$.

$$S_2 = \{\emptyset, 4, 5, 6, 13, 23, 234, 135, 1236, 456\}$$

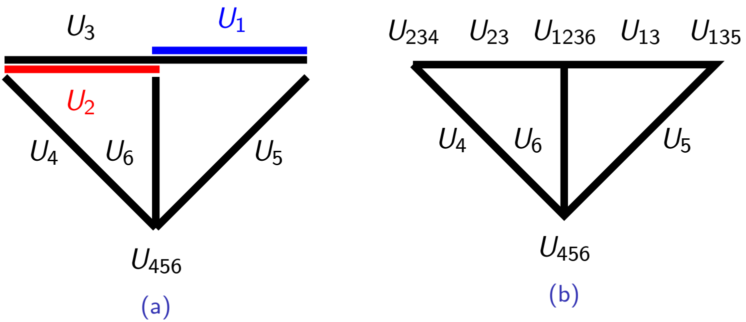


Figure: Receptive field setup (a) and realization (b) of S_2

Closed convexity of sunflowers.

The realization for $n \geq 3$ is drawn as follows

1. Draw a $(2^n - 2)$-sided, regular polygon.
   This polygon is the receptive field for the codeword $1 \cdots n(n+1)(2n+2)$.
2. Draw the circle that passes through the vertices of the polygon.

   The circle is U_{n+1}.
   The clopen subset of the circle outside of one of the edges of the polygon corresponds to one of the nonempty proper subsets of $[n]$.
3. Pick a point in a plane parallel to the one in which the polygon sits and let $U_{2n+2} = conv\{point, vertices\}$.
4. Draw a line segment from each subset of the circle $U_{1 \cdots (i-1)(i+1)(n+1)}$ for $1 \leq i \leq n$ to the point from (3). This line is U_{n+1+i}.

Sunflower codes (Jeffs [3])

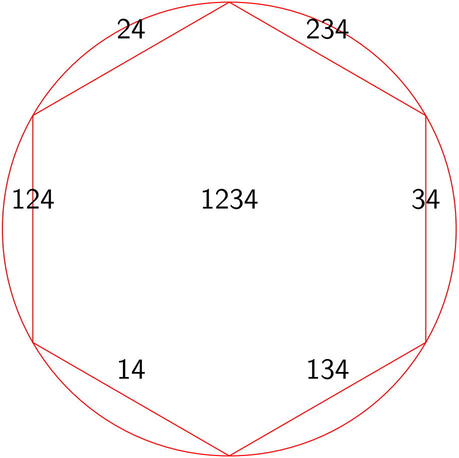


Figure: Face of S_3

$$S_3 = \{\emptyset, 5, 6, 7, 8, 14, 24, 124, 34, 134, 234,$$
$$2345, 1247, 1346, 12348, 5678\}$$

Sunflower codes (Jeffs [3])

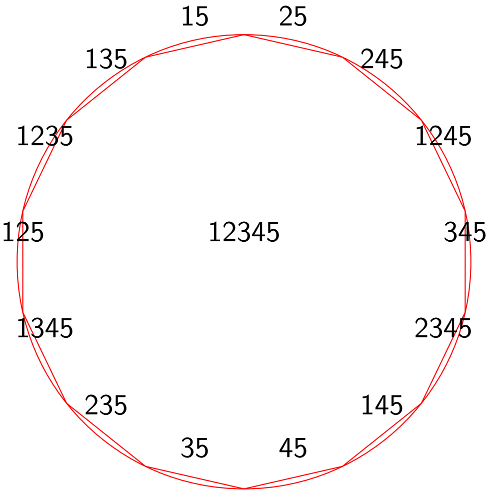


Figure: Face of S_4 based on L28 from Curto et al. [4]

$S_4 = \{\emptyset, 6, 7, 8, 9, 10, 15, 25, 125, 35, 135, 235, 1235,$

45, 145, 245, 1245, 345, 1345, 2345, 13457, 12359

References

1. Curto 2018 C. Curto, E. Gross, J. Jeffries, K. Morrison, Z. Rosen, A. Shiu, N. Youngs, *Algebraic signatures of convex and non-convex codes*, Journal of Pure and Applied Algebra, **223** (2018), no. 9, 3919-3940.
2. Goldrup 2019 S. A. Goldrup, K. Phillipson, *Classification of open and closed convex codes on five neurons*, Advances in Appl. Math., **112** (2020).
3. Amzi R. Amzi Jeffs, *Sunflowers of convex open sets*, Advances in Applied Mathematics, **111** 2019.
4. Curto 2019 C. Curto, E. Gross, J. Jeffries, K. Morrison, M. Omark, Z. Rosen, A. Shiu, N. Youngs, *What makes a neural code convex?*, Journal of Appl. Algebra Geometry, **1** (2020), 222-238.

Thank you!

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