# Calculating the Correlation Kernel along Space-Like Paths

Lauren Neudorf and Emily Bosche

Texas A&M University

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# Background

## Symplectic Group

$$Sp_{2n} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (1)$$

- Lie Algebra
- Irreducible Representation
- $A = -D^T, B = B^T, C = C^T$
- Parametrized by $\{(\lambda_1, ..., \lambda_n) : \lambda_1 \geq ... \geq \lambda_n \geq 0\}$
    - Where $\lambda = (\lambda_1, ..., \lambda_n)$ are mapped onto $x = (x_1, ..., x_n)$ where $x_1 > ... > x_n \geq 0$ and $x_i = \lambda_i + n - i$

# Review of Markov Processes

## Markov Chain and Process

-The defining characteristic of a Markov chain is that no matter how the process arrived at its present state, the possible future states are fixed.
-The continuous time version of a Markov Chain is a Markov Process.
-A Markov Process at a fixed time is called Determinantal

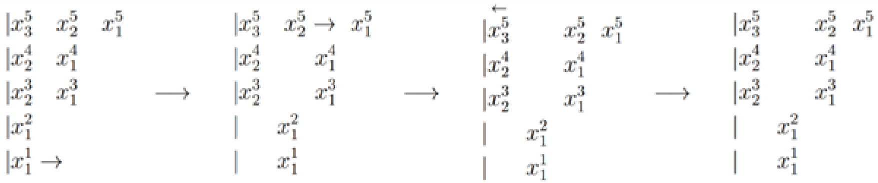
## Transition Matrices

P(t), $t \geq 0$. The entries are the probabilities, $p_{x,y}(t)$, to transition from one state (x) to another (y) where $p_{x,y}(t) = P(X_{t+1} = y | X_t = x)$

# Push-Block Model [Cerenzia '18]

## Barriers and State Space

- $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{+}$ integers
- $X_{i+1}^{(K+1)} < X_i^{(K)} \leq X_i^{(K+1)}$ for odd values of K
- $X_{i+1}^{(K+1)} \leq X_i^{(K)} < X_i^{(K+1)}$ for even values of K



# Introduction of Problem

## Our Correlation Kernel Equation

$$K^{(t)}((s_1, n_1), (s_2, n_2)) =$$

$$1_{(n_1 \geq n_2)} \cdot \frac{2^{a_{n_1}+1/2}}{\pi} \int_{-1}^{1} J_{s_1,a_{n_1}}(x) J_{s_2,a_{n_2}}(x) \cdot (1-x)^{r_{n_1}-r_{n_1}+a_{n_1}} (1+x)^{1/2} dx$$

$$+ \frac{2^{a_{n_1}+1/2}}{\pi} \int_{-1}^{1} \oint \frac{e^{t(x-1)}}{e^{t(x-1)}} J_{s_1,a_{n_1}}(x) J_{s_2,a_{n_2}}(u) \cdot \frac{(1-x)^{r_{n_1}+a_{n_1}} (1+x)^{1/2}}{(1-u)^{r_{n_2}} (x-u)} du dx$$

- Our Problem: Write $K(\cdot, \cdot)$ for $(x_i, n_i, t_i)$ where $1 \leq i \leq K, t_1 \leq ... \leq t_k, n_1 \geq ... \geq n_k$
- Parameters
  - $(s_i, n_i) \in Z_{\geq 0} \times Z_+$
  - $a_n = 1/2$ if n is even
    $a_n = -1/2$ if n is odd
  - $r_n = \left\lfloor \frac{n+1}{2} \right\rfloor$, represents the number of particles on the $n^{th}$ level
  - $J_{s,\pm 1/2}(x)$ represents a Jacobi polynomial
  - $\int_{-1}^{1} J_{s_1,\pm 1/2}(x) J_{s_2,\pm 1/2}(x) (1-x)^{\pm 1/2} (1+x)^{1/2} dx$

# Cerenzia '18 and Zhou '21

## Definition 1.1 (Cerenzia)

$$\tau^{n,a}_{t_j^{2n-1/2+a},t_i^{2n-1/2+a}} = \tau^{n,a}_{t_i^{2n-1/2+a},t_{i-1}^{2n-1/2+a}} * \ldots * \tau^{n,a}_{t_j^{2n-1/2+a},t_{j-1}^{2n-1/2+a}}$$

## Definition 1.2(Cerenzia)

$$\phi^{t_{b_1}^{2n_1-1/2+a_1},t_{b_2}^{2n_2-1/2+a_2}} =$$

$$\tau^{n_2,a_2}_{t_{c(2n_2+a_2+1/2)}^{2n_2+a_2+1/2},t_{b_2}^{2n_2}} * \phi^{n_2,a_2}_y * \tau^y_{t^y_{c(y)},t_0} * \ldots * \phi^m_{n_1,a_1} * \tau^{n_1,a_1}_{t_{b_1}^{2n_1+a_1+1/2},t_0^{n_1,a_1}}$$

# Cerenzia '18 and Zhou '21 (cont.)

## Definition 1.3 (Zhou)

$$const \times \prod_{n=1}^{N}[det[\phi_{n-1,+}^{n,-}(x_\ell^{n,-}(t_{c(2n-1)}^{2n-1}), x_K^{n-1,+}(t_0^{2n-2}))]_{1\leq K,\ell\leq n}$$

$$\times \prod_{b=1}^{c(2n-1)} det[\tau_{t_b^{2n-1},t_{b-1}^{2n-1}}^{n,-}(x_\ell^{n,-}(t_b^{2n-1}), x_K^{n,-}(t_{b-1}^{2n-1}))]_{1\leq K,\ell\leq n}$$

$$\times det[\phi_{n,-}^{n,+}(x_\ell^{n,+}(t_{c(2n)}^{2n}), x_K^{n,-}(t_0^{2n-1}))]_1 \leq K,\ell \leq n$$

$$\times \prod_{b=1}^{c(2n)} det[\tau_{t_b^{2n},t_{b-1}^{2n}}^{n,+}(x_\ell^{n,+}(t_b^{2n}), x_K^{n,+}(t_{b-1}^{2n}))]_{1\leq K,\ell\leq n}]$$

$$\times det[_{N-\ell}^{N,a}(x_K^{2N-1/2+a}(t_0^{2N-1/2+a}))]_{1\leq K,\ell\leq N}$$

- Particle positions at time $t$ are defined by $X_K^{n,a}$

# Cerenzia '18 and Zhou '21 (cont.)

**Result**

$$const \times \prod_{n=1}^{N} [det[\phi^{n_1,-}_{n_1-1,+}(x_\ell^{n,-}(t_j^{2n_1-1}), x_K^{n_1-1,+}(t_i^{2n_1-1}))]_{1\leq K,\ell\leq n}$$
$$\times \prod_{b_1=1}^{c(2n_1-1)} det[\tau^{n_1,-}_{t_{b_1}^{2n_1-1}, t_{b_1-1}^{2n_1-1}}(x_\ell^{n_1,-}(t_{b_1}^{2n_1-1}), x_K^{n_1,-}(t_{b_1-1}^{2n_1-1}))]_{1\leq K,\ell\leq n}$$
$$\times det[\phi^{n_2,+}_{n_2,-}(x_\ell^{n_2,+}(t_j^{2n_2}), x_K^{n_2,-}(t_i^{2n_2-1}))]_1 \leq K, \ell \leq n$$
$$\times \prod_{b_2=1}^{c(2n_2)} det[\tau^{n_2,+}_{t_{b_2}^{2n_2}, t_{b_2-1}^{2n_2}}(x_\ell^{n_2,+}(t_{b_2}^{2n_2}), x_K^{n_2,+}(t_{b_2-1}^{2n_2}))]_{1\leq K,\ell\leq n}]$$
$$\times det[^{N,a}_{N-\ell}(x_K^{2N-1/2+a}(t_i^{2N-1/2+a}))]_{1\leq K,\ell\leq N}$$

# Equations 33 and 34

$\int_{-1}^{1} J_k^{a,b}(x) J_\ell^{a,b}(x) \cdot (1-x)^a (1+x)^b dx$

- $x = \frac{z+z^{-1}}{2}$ for all $x \in [-1, 1]$
- $J_{k,1/2}(\frac{z+z^{-1}}{2}) = \frac{z^{k+1} - z^{-(k+1)}}{z - z^{-1}}$
- $J_{k,-1/2}(\frac{z+z^{-1}}{2}) = \frac{z^{k+(1/2)} + z^{-(k+(1/2))}}{z^{1/2} + z^{-1/2}}$

## Equation 33

$$\langle f, g \rangle_a := \frac{2^{a+(1/2)}}{\pi} \int_{\mathbb{R}} f(x) g(x) w_{(a,1/2)}(x) dx$$

## Equation 34

$$T(x) = \sum_{k=0}^{\infty} \langle J_{k,a_n}, T \rangle_{a_n} J_{k,a_n}(x)$$

# Application of Equation 33

- $\left.\begin{matrix} \tau^{n,-}_{t_b^{2n-1}, t_{b-1}^{2n-1}} \\ \tau^{n,+}_{t_b^{2n}, t_{b-1}^{2n}} \end{matrix}\right\} \tau^{n,a}_{t_1,t_2}(x,y) = \left\langle J_{x,a}, J_{y,a}\varphi^{t_1-t_2} \right\rangle_a$
- where $\varphi^{t_1,t_2} \to \varphi^t(x) = e^{t(x-1)}$
- $f = J_{x,a}$ and $g = J_{y,a}\varphi^{t_1-t_2}$
- $\langle f, g\rangle_a := \frac{2^{a+(1/2)}}{\pi} \int f(x)g(x)w_{(a,1/2)}(x)dx$

**Result**

$$\frac{2^{a+1/2}}{\pi} \int_{\mathbb{R}} J_{x,a}(x)J_{y,a}\varphi^{t_1-t_2}(x)w_{(a,1/2)}(x)dx$$

# Lemma 2.2

**Lemma**

*for* $a = \pm 1/2$*,* $b = 1/2$*,* $-1 \leq \zeta \leq 1$*, with Test Function* $T \in C^1$*[-1,1], then*

$T(\zeta) = \sum_{k=0}^{\infty} \int_{-1}^{1} \frac{J_k^{a,1/2}(x) J_k^{a,1/2}(\zeta)}{h_k^{a,1/2}} T(x)(1-x)^a (1+x)^{1/2} dx$

# Background Needed for Proof of Analog

### [11] from Kuan '11 and Equation 32 and 33 from Cerenzia '18

$a = \pm 1/2,\ b = 1/2$

$h_k^{(a,b)} = \frac{\pi c_k^2}{W^{(a,b)}(k)}$

Where $W^{(a,b)}(k) = \begin{cases} 2 & \text{if } a = b = 1/2 \\ 1 & \text{if } a = -1/2,\ b = 1/2 \end{cases}$

### (4.1.7) and (4.1.8)from Szegö

$P_n^{(-1/2,1/2)}(x) = \frac{1\times3\times5...(2n-1)}{2\times4\times6...2n} \frac{\cos((2n+1)(\phi/2))}{\cos((\phi/2))}$

$P_n^{(1/2,1/2)}(x) = 2\frac{1\times3\times5...(2n+1)}{2\times4\times6...(2n+2)} \frac{\sin(\phi(n+1))}{\sin\phi}$

where $x = cos\phi$

# Analog of Lemma 2.2 (cont.)

**Proof.**

Let a=1/2, b=1/2, $x = \cos\phi$, $\zeta = \cos\theta$, utilizing (4.1.7) of Szegö and [11] of Kuan:
$\frac{J_k^{(1/2,1/2)}(x) J_k^{(1/2,1/2)(\zeta)}}{h_k^{(1/2,1/2)}} = (\frac{2}{\pi})(\frac{\sin((k+1)\phi)}{\sin(\phi)})(\frac{\sin((k+1)\theta)}{\sin(\theta)})$ and
$w = (1-x)^{1/2}(1+x)^{1/2} = \sin\phi$ □

# Analog of Lemma 2.2 (cont.)

Proof.

Since T is $C^1$, the Fourier series of T converges to T.

$T(\cos(\phi)) = \sum_{k=0}^{\infty} \hat{T}_k \frac{\sin(k\phi)}{sin\phi} = \hat{T}_0 + \hat{T}_1 \frac{\sin(2\phi)}{sin\phi} + \ldots,$

Where $\hat{T}_k = \frac{2}{\pi} \int_0^{\pi} T(\cos\phi) \frac{\sin((k+1)\phi)}{\sin\phi} d\phi$ □

# Lemma 2.2 Proof (cont.)

**Proof.**

Therefore combining the above steps gives:

$$\sum_{k=0}^{\infty} \int_{-1}^{1} \frac{J_k^{1/2,1/2}(x) J_k^{1/2,1/2}(\zeta)}{h_k^{1/2,1/2}} T(x)(1-x)^{1/2}(1+x)^{1/2} dx =$$

$$\frac{2}{\pi} \sum_{k=1}^{\infty} \left( \frac{\sin((k+1)\theta)}{\sin\theta} \int_0^{\pi} T(\cos\phi) \frac{\sin((k+1)\phi)}{\sin\phi} d\phi \right) = \hat{T}_0 + \hat{T}_1 \frac{\sin(2\theta)}{sin\theta} + \ldots, = T(\cos\theta) = T(\zeta)$$

□

# Proof of Lemma 2.2 (cont)

**Proof.**

For the case of $a = -1/2$, using the same (4.1.8) of Szegö and [11] from Kuan as used above, you get $\frac{J_k^{(-1/2,1/2)}(x) J_k^{(-1/2,1/2)}(\zeta)}{h_k^{(-1/2,1/2)}} = \frac{1}{\pi} \frac{\cos((2n+1)(\phi/2))}{\cos((\phi/2))} \frac{\cos((2n+1)(\theta/2))}{\cos((\theta/2))}$, and the rest of this case follows similarly to a=1/2 □

# Proposition 4.2 [Cerenzia '18]

## Define

For any $(s, n), (t, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}$, define the functions:

- $\Phi^{m}_{r_m - k}(t) := \frac{1}{2\pi i} \oint \frac{J_{t,\alpha_m}(w)}{E(w)(w-1)^{r_m-k+1}dw}$
- $\phi^{[n,m)}(s, t) := -\frac{1}{2\pi i} \oint \left\langle J_{s,\alpha_n}, \frac{J_{t,\alpha_m}(u)(u-1)^{r_n-r_m}}{x-u} \right\rangle_{\alpha_n} du$, for $n < m$

## The Determinantal Correlation function with Kernel

$K^{w}((s, n), (t, m)) = -\Phi^{[n,m)}(s, t)1_{(n<m)} + \sum_{k=1}^{r_m} \Psi^{n}_{r_n-k}(s)\Phi^{m}_{r_m-t}(t)$

# What Would Be Next!

- Use Proposition 4.2, using $\phi$ to find $\Psi$, aiming to get the closed from of $\phi$
- Use Lemma 2.2 to help simplify the integral results from Equation 33
- Determine if Probability $((s_i, n_i)$ is occupied at time $t_i$ for $1 \leq i \leq k)$ $= det[\psi]det[\tau]det[\phi]$

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# References


- Cerenzia, M. A path property of Dyson gaps, Plancherel measures for Sp($\infty$), and random surface growth. arXiv:1506.08742
- Borodin, A  Kuan, J. Random surface growth with a wall and Plancherel measures for O($\infty$). Communications on Pure and Applied Mathematics, Volume 63, Issue 7, pages 831-894, July 2010. arXiv:0904.2607
- Szegö, G. Orthogonal Polynomials. American Mathematical Society, Prov- idence, 1975.