Algebraic properties of values of newform Dedekind sums

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Section 1

Dirichlet Characters

A Dirichlet character (modulo q) $\chi : \mathbb{Z} \to \mathbb{C}^\times$ is a mapping with the following properties:

- $\chi(ab) = \chi(a)\chi(b)$.
- $\chi(a) = \begin{cases} = 0 & gcd(a,q) \neq 1 \\ \neq 0 & gcd(a,q) = 1. \end{cases}$
- $\chi(a \pm q) = \chi(a)$.

These properties imply that when $(a, q) = 1$, $\chi(a)$ is a $\phi(q)^{th}$ roots of unity.

$SL_2(\mathbb{Z})$ and some subgroups

Recall the following definitions:

Definition

$$SL_2(\mathbb{Z}) = \{\gamma \in M_2(\mathbb{Z}) |\, det(\gamma) = 1\}$$

$$\Gamma_0(N) = \{\gamma \in SL_2(\mathbb{Z}) | \gamma \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \text{(mod } N)$$

$$\Gamma_1(N) = \{\gamma \in SL_2(\mathbb{Z}) | \gamma \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \text{(mod } N)$$

$$\Gamma(N) = \{\gamma \in SL_2(\mathbb{Z}) | \gamma \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{(mod } N)\}$$

Remark

$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z})$.
$\Gamma(N)$ *is the kernel of the reduction map* $SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})$.

Modular Forms

A modular form f of weight k is a function on $\mathbb{H}$ that has a symmetry with the group action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ by linear fractional transformations. Specifically:

- $f(\begin{bmatrix} a & b \\ c & d \end{bmatrix} z) = f(\frac{az+b}{cz+b}) = (cz+d)^k f(z)$ for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.
- f is holomorphic/complex analytic.
- $f(z)$ is bounded as $Im(z) \longrightarrow \infty$.

We can extend this concept to what are called automorphic forms by relaxing the holomorphicity requirement and including an automorphy factor ϵ in the symmetry:

$$f(\begin{bmatrix} a & b \\ c & d \end{bmatrix} z) = f(\frac{az+b}{cz+b}) = \epsilon(a,b,c,d)(cz+d)^k f(z).$$

Section 2

Eisenstein Series

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  $\left(\text{Note } \psi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \psi(d)\right)$.

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  $\left(\text{Note } \psi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \psi(d)\right)$.
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  $\left(\text{Note } \psi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \psi(d)\right)$.
- Eigenfunction of all Hecke operators T_n with eigen value $\lambda_{\chi_1,\chi_2}(n,s)$.
- $E^*_{\chi_1,\chi_2}(z,s)$ (the completed Eisenstein series) at $s = 1$ decomposes into holomorphic and anti-holomorphic parts $f_{\chi_1,\chi_2}(z) + \chi_2(-1)\overline{f}_{\overline{\chi_1},\overline{\chi_2}}(z)$.

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where

$$\lambda_{\chi_1,\chi_2}(n,s) = \chi_2(\mathsf{sgn}(n)) \sum_{ad=|n|} \chi_1(a)\overline{\chi_2}(b) \left(\frac{b}{a}\right)^{s-\frac{1}{2}}.$$

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- The first definition of the newform Dedekind sum is as follows:

Definition

For primitive χ_1, χ_2 mod q_1, q_2 where $\chi_1\chi_2(-1) = 1$, and $\gamma \in \Gamma_0(q_1q_2)$

$$S_{\chi_1\chi_2}(\gamma) = \frac{\tau(\overline{\chi_1})}{\pi i}(f_{\chi_1,\chi_2}(\gamma z) - \psi(\gamma) f_{\chi_1,\chi_2}(z))$$

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$$S_{\chi_1\chi_2}(\gamma) = S_{\chi_1\chi_2}(a,c) = \sum_{j \bmod c} \sum_{n \bmod q_1} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right)$$

where

$$B_1(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2} & \text{otherwise.} \end{cases}$$

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- From the above theorem, we see the sum always lies in $\mathbb{Q}(\zeta_n)$ ($\mathbb{Q}$ adjoined with some root of unity ζ_n).
- $S_{\chi_1\chi_2}$ is never trivial.
- $S_{\chi_1\chi_2}(\gamma_1\gamma_2) = S_{\chi_1\chi_2}(\gamma_1) + \psi(\gamma_1)S_{\chi_1\chi_2}(\gamma_2)$.
- $S_{\chi_1\chi_2}$ is a homomorphism when $\psi = \mathbf{1}$, while $S_{\chi_1\chi_2}|_{\Gamma_1(q_1q_2)}$ is always a homomorphism.

Section 3

The Hecke operator T_n

Definition

The weight 0 Hecke operator on automorphic forms with central character ψ is

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As noted, $E_{\chi_1,\chi_2}(z,s)$ is an eigen function for this family of commuting linear operators with eigenvalue $\lambda_{\chi_1,\chi_2}(n,s)$. When we specialize to $s=1$ we deduce that $f_{\chi_1,\chi_2}(z)$ is as well:

$$\begin{aligned} T_n f_{\chi_1,\chi_2}(z) &= \frac{1}{\sqrt{n}} \sum_{ad=n} \psi(n) \sum_{b \text{ (mod d)}} f_{\chi_1,\chi_2}(\frac{az+b}{d}) \\ &= \lambda_{\chi_1,\chi_2}(n,1) f_{\chi_1,\chi_2}(z). \end{aligned}$$

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This gives an easy definition of $T_n S_{\chi_1\chi_2}$:

$$\begin{aligned} T_n S_{\chi_1\chi_2}(\gamma) &= \frac{\tau(\overline{\chi_1})}{\pi i}(T_n f_{\chi_1,\chi_2}(\gamma z) - \psi(\gamma) T_n f_{\chi_1,\chi_2}(z)) \\ &= \lambda_{\chi_1,\chi_2}(n,1) S_{\chi_1\chi_2}(\gamma). \end{aligned}$$

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$$= \lambda_{\chi_1,\chi_2}(n,1) S_{\chi_1\chi_2}(\gamma).$$

We can then use the fourier expansion of $f_{\chi_1,\chi_2}(z)$ and the z independence of $S_{\chi_1\chi_2}(\gamma)$ to get the following identity:

Theorem

For $h, k, n \in \mathbb{Z}$, $q_1 q_2 | k$, and $n, k > 0$

$$\frac{1}{\sqrt{n}} \sum_{ad=n} \chi_1\overline{\chi_2}(a) \sum_{b \text{(mod d)}} S_{\chi_1\chi_2}(ah+bk, dk) = \lambda_{\chi_1,\chi_2}(n,1) S_{\chi_1\chi_2}(h,k).$$

Takeaways

This is a generalization of the classical case due to M. Knopp:

Theorem

For $h, k, n \in \mathbb{Z}$, $k, n > 0$

$$\sum_{ad=n} \sum_{b \pmod{d}} s(ah+bk, dk) = \sigma(n) s(h,k), \quad \sigma(n) = \sum_{d|n} d.$$

More importantly, the fact that the Dedekind sums are eigenvectors of a family of commuting linear operators means they are linearly independent.

Section 4

Definition of Galois action

Definition

Let $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Then,

$$\begin{aligned}\sigma S_{\chi_1\chi_2}(\gamma) &= \sigma\Big(\sum_{j \bmod c} \sum_{n \bmod q_1} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right)\Big) \\ &= \sum_{j \bmod c} \sum_{n \bmod q_1} \overline{\chi_2^\sigma}(j)\overline{\chi_2^\sigma}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right) \\ &= S_{\chi_1^\sigma,\chi_2^\sigma}(\gamma)\end{aligned}$$

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The Dedekind sum $S_{\chi_1\chi_2}$ takes values in the number field F if and only if χ_1 and χ_2 take values in F.

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The backward direction of this theorem comes directly from the finite sum definition. For the other, we choose any $\sigma \in Gal(\overline{\mathbb{Q}}/F)$ and must have

$$S_{\chi_1\chi_2} = S_{\chi_1^\sigma,\chi_2^\sigma}.$$

The linear independence of the sums then gives $\chi_1 = \chi_1^\sigma$ and $\chi_2 = \chi_2^\sigma$, proving the result.

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Example

$S_{\chi_1\chi_2}$ takes rational values if and only if χ_1 and χ_2 are rational characters.

Structure of $\Gamma_1(q_1q_2)/K^1_{\chi_1,\chi_2}$

We use the following definitions:

- $K_{\chi_1,\chi_2} = \{\gamma \in \Gamma_0(q_1q_2) | S_{\chi_1\chi_2}(\gamma) = 0\}$
- $K^1_{\chi_1,\chi_2} = \Gamma_1(q_1q_2) \cap K_{\chi_1,\chi_2}$
- F is the smallest number field over $\mathbb{Q}$ in which $S_{\chi_1\chi_2}$ takes values.

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We will investigate $\Gamma_1(q_1q_2)/K^1_{\chi_1,\chi_2} \cong S_{\chi_1\chi_2}(\Gamma_1(q_1q_2))$ because $S_{\chi_1\chi_2}|_{\Gamma_1(q_1q_2)}$ is a homomorphism, and we will show its rank is equal to the degree of F over $\mathbb{Q}$. These arguments are carried over without changes to $\Gamma_0(q_1q_2)/K_{\chi_1,\chi_2}$ in the case of $\chi_1\overline{\chi_2} = \mathbf{1}$.

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- Since $\Gamma_1(q_1q_2)$ is of finite index in $SL_2(\mathbb{Z}) = \langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rangle$, it must also have a set of finite generators $\{\gamma_j\}_{j=1}^r$. Then $\{S_{\chi_1\chi_2}(\gamma_j)\}_{j=1}^r$ generates $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2))$.

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- We also know $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2)) \subset (\mathbb{C}, +)$ implies $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2))$ is torsion free.

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- We also know $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2)) \subset (\mathbb{C}, +)$ implies $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2))$ is torsion free.
- Combining these two facts shows $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2))$ is a free abelian group by the structure theorem of ableian groups.

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- Since F is the fraction field of its algebraic integers ($\mathcal{O}_F$), we may "bound" the denominators of $S_{\chi_1\chi_2}(\gamma_j)$ by some $d \in F$.

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- Then $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2)) \subset \frac{1}{d}\mathcal{O}_F$
- $\mathcal{O}_F$ (and its fractional ideals) are free abelian groups of rank $[F : \mathbb{Q}]$.

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- Suppose $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2)) = \bigoplus_{i=1}^{d} \alpha_i\mathbb{Z}$ for $\alpha_i \in F$, $d < [F : \mathbb{Q}] = n$.

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- Then consider the n distinct Dedekind sums $S_{\chi_1^{\sigma_j}\chi_2^{\sigma_j}}$ for $\sigma_j \in Gal(F/\mathbb{Q})$.
- Then we can construct a $d \times n$ matrix $(\alpha_i^{\sigma_j})_{ij}$, that, by its dimension, has nontrivial kernel, contradicting the linear independence of the Dedekind sums.

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