# Higher Moments of Newform Dedekind Sums

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# Motivation

**Definition**

A partition, $\lambda$, of a natural number $n$ is a finite non-increasing sequence of positive integers $b_1, b_2, \ldots, b_r$ such that $\sum_{i=1}^{r} b_i = n$.

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Example: Partitions of 5

$$5, \quad 4+1, \quad 3+2, \quad 3+1+1, \quad 2+2+1,$$
$$2+1+1+1, \quad 1+1+1+1+1$$

so $p(5) = 7$.

# Motivation

The generating function for $p(n)$ is given by

$$\sum_{n\geq 0} p(n)q^n.$$

We can relate this to modular form known as the Dedekind eta function by the equation

$$\sum_{n\geq 0} p(n)q^n = \frac{q^{1/24}}{\eta(z)},$$

where $z \in \mathbb{H}$, $q = e^{2\pi i z}$, and

$$\eta(z) := q^{\frac{1}{24}} \prod_{n\geq 1} (1 - q^n).$$

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$$\Longleftrightarrow$$
$$f(\gamma + x) = f(x) \text{ for all } \gamma \in r\mathbb{Z}, x \in \mathbb{R}.$$

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Analogously, a discrete subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by linear fractional transformations and

$$f : \mathbb{H} \to \mathbb{C} \text{ is a modular form}$$

$$\iff$$

$$f(\gamma z) = \epsilon(a, b, c, d)(cz + d)^k f(z) \text{ for all } \gamma \in \Gamma, z \in \mathbb{H}$$

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(plus a holomorphicity and growth condition on $f$).

# Motivation

Using this relation between $p(n)$ and $\eta(z)$ along with the modularity properties of $\eta(z)$, Hardy and Ramanujan proved that $p(n)$ is approximately

$$\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}$$

for large $n$.

# Motivation

More specifically, the Dedekind eta function is a modular form of weight 1/2 and satisfies the transformation law

$$\eta\left(\frac{az+b}{cz+d}\right) = \epsilon(a,b,c,d)(cz+d)^{\frac{1}{2}}\eta(z),$$

where

$$\epsilon(a,b,c,d) := \begin{cases} e^{\frac{bi\pi}{12}}, & c=0, d=1 \\ e^{i\pi\left(\frac{a+d}{12c} - s(d,c) - \frac{1}{4}\right)}, & c>0. \end{cases}$$

# Background

Our research involves a certain generalization of the classical Dedekind sum which also arises from a modular form. We call this generalization the newform Dedekind sum and denote it by $S_{\chi_1,\chi_2}(a, c)$.

# Background

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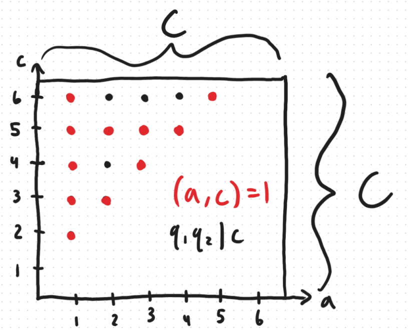
A lot of work has been done on the value distribution of the classical Dedekind sum. This includes questions about how frequently $s(h,k)$ can be large, for instance. On the other hand, little is known about the value distribution of $S_{\chi_1,\chi_2}(a,c)$.

# Background

Corbett and Young recently studied the frequency of large values of the newform Dedekind sum. For $C, \alpha > 1$, define

$$F(\alpha, C) := \#\{(a,c) \in S : |S_{\chi_1,\chi_2}(a,c)| > \alpha \log^3 C\},$$

where $S$ is the collection of points



# Background

**Theorem (Corbett, Young)**

*For large $C$,*

$$F(\alpha, C) \ll_{\chi_1,\chi_2} \frac{C^2}{\alpha} + C^2 \frac{\log\log C}{\log C}. \tag{1}$$

The value of $F(\alpha, C)$ depends heavily on $\alpha$, but if $\alpha \gg \frac{\log C}{\log\log C}$, then (1) becomes

$$F(\alpha, C) \ll_{\chi_1,\chi_2} C^2 \frac{\log\log C}{\log C}, \tag{2}$$

which is independent of $\alpha$. This suggests that this theorem is not particularly effective for large values of $\alpha$.

# Background

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$$\sum_{\substack{a \pmod{c} \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^{2r}.$$

# Definition of $S_{\chi_1,\chi_2}(a,c)$

We now formally introduce the newform Dedekind sum $S_{\chi_1,\chi_2}(a,c)$. Let $\chi_1, \chi_2$ be primitive Dirichlet characters modulo $q_1, q_2$, respectively, with $\chi_1\chi_2(-1) = 1$. For $z \in \mathbb{H}$, define

$$f_{\chi_1,\chi_2}(z) := \sum_{\ell \geq 1} \sum_{k \geq 1} \frac{\chi_1(\ell)\overline{\chi_2}(k)}{\ell} e(k\ell z) \tag{3}$$

and

$$\phi_{\chi_1,\chi_2}(\gamma, z) := f_{\chi_1,\chi_2}(\gamma z) - \psi(\gamma) f_{\chi_1,\chi_2}(z), \tag{4}$$

where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(q_1 q_2)$, $\psi := \chi_1 \overline{\chi_2}$, and $\psi(\gamma) := \psi(d)$.

# Definition of $S_{\chi_1,\chi_2}(a,c)$

By lemma 2.1 in [SVY20], $\phi_{\chi_1,\chi_2}$ is independent of $z \in \mathbb{H}$, which allows us to define the newform Dedekind sum as follows.

**Definition**

[SVY20] Let $\chi_1, \chi_2$ be nontrivial primitive Dirichlet characters modulo $q_1, q_2$, respectively, with $\chi_1\chi_2(-1) = 1$. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(q_1q_2)$ with $c \geq 1$. For any $z \in \mathbb{H}$, the newform Dedekind sum is defined by

$$S_{\chi_1,\chi_2}(a,c) := \frac{\tau(\overline{\chi_1})}{\pi i}\phi_{\chi_1,\chi_2}(\gamma, z), \tag{5}$$

where $\tau(\overline{\chi_1})$ is the Gauss sum given by $\tau(\overline{\chi_1}) := \sum_{n=1}^{q_1} \overline{\chi_1}(n)e_{q_1}(n)$.

# Definition of $S_{\chi_1,\chi_2}(a,c)$

As shown in [SVY20], we may alternatively define the newform Dedekind sum by the equation

$$S_{\chi_1,\chi_2}(a,c) = \sum_{j \pmod{c}} \sum_{n \pmod{q_1}} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right), \tag{6}$$

where

$$B_1(x) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & x \in \mathbb{R} \setminus \mathbb{Z} \\ 0, & x \in \mathbb{Z} \end{cases}$$

denotes the first Bernoulli function.

# Prior Work

In [DG19], Dillon and Gaston computed the finite Fourier transform of (6) to obtain an asymptotic formula for the second moment of $S_{\chi_1,\chi_2}(a,c)$.

**Theorem (Dillon, Gaston)**

*Let $\chi_1, \chi_2$ be nontrivial primitive Dirichlet characters modulo $q_1, q_2$, respectively, with $\chi_1\chi_2(-1) = 1$, and let $q_1 q_2 \mid c$. Then*

$$\sum_{\substack{a \pmod{c} \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^2 = q_1 c^{2+o(1)}. \tag{7}$$

As remarked in [CY23], (7) shows that $F(\alpha, C) \ll \alpha^{-2} C^{3+o(1)}$, which is better than (2) for $\alpha \gg C^{1/2+o(1)}$.

# Prior Work

We now discuss the problem of bounding higher moments of $S_{\chi_1,\chi_2}(a,c)$ from above. Using (6), we obtain the trivial bound $|S_{\chi_1,\chi_2}(a,c)| \leq q_1 c$, which implies

$$\sum_{\substack{a \pmod c \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^{2r} \ll q_1^{2r} c^{2r+1}. \tag{8}$$

# Prior Work

On the other hand, Corbett and Young showed in [CY23] that $S_{\chi_1,\chi_2}(a,c)$ can have size proportional to $\sqrt{\frac{q_1}{q_2}}c$. This shows that, at best,

$$\sum_{\substack{a \pmod c \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^{2r} \ll_r (q_1/q_2)^r\, c^{2r}. \tag{9}$$

We obtain a result in between (8) and (9).

# Results

**Theorem**

*Let $\chi_1, \chi_2$ be nontrivial primitive Dirichlet characters modulo $q_1, q_2$, respectively, with $\chi_1\chi_2(-1) = 1$, and let $q_1 q_2 \mid c$. For $r \geq 1$, we have*

$$\sum_{\substack{a \pmod{c} \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^{2r} \ll_r q_1^r c^{2r} \log^{2r} c.$$

In the case when $r = 1$, this yields a slightly sharper upper bound than given in (7).

# Results

As an immediate corollary of our theorem, we obtain a bound on the number of values of $a \pmod{c}$ for which $|S_{\chi_1,\chi_2}(a,c)| \geq m$.

## Corollary

*If* $m > 0$ *and*

$$N_c(m) := \#\{a \pmod{c} : |S_{\chi_1,\chi_2}(a,c)| \geq m\},$$

*then*

$$N_c(m) \ll_r \frac{q_1^r c^{2r} \log^{2r} c}{m^{2r}}. \tag{10}$$

We can see this by applying our bound and rearranging

$$\sum_{\substack{a \pmod{c} \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^{2r} \geq \sum_{\substack{a \pmod{c} \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^{2r} \geq m^{2r} N_c(m).$$

We note that (10) is better than the trivial bound $N_c(m) \leq c$ whenever

$$m \gg_r \sqrt{q_1} c^{1-\frac{1}{2r}} \log c.$$

That is, our bound works best for large values of $m \leq q_1 c$.

# Results

Finally, we obtain result complementary to Corbett and Young's theorem that provides a stronger bound on $F(\alpha, C)$ for large values of $\alpha$.

**Corollary**

*We have*

$$F(\alpha, C) \ll_{r,\chi_1,\chi_2} \frac{C^{2r+1}}{\alpha^{2r} \log^{4r} C},$$

*which is stronger than* (2) *whenever*

$$\alpha \gg_{r,\chi_1,\chi_2} \frac{C^{1-\frac{1}{2r}}}{(\log\log C)^{\frac{1}{2r}} \log^{2-\frac{1}{2r}} C}.$$

# Results

This can quickly be seen by applying (10) to get

$$F(\alpha, C) = \sum_{\substack{c \leq C \\ q_1 q_2 | c}} N_c(\alpha \log^3 C) \ll_r \frac{C^{2r+1}}{\alpha^{2r} \log^{4r} C},$$

which is better than (2) if and only if

$$\alpha \gg_{r,\chi_1,\chi_2} \frac{C^{1-\frac{1}{2r}}}{(\log \log C)^{\frac{1}{2r}} \log^{2-\frac{1}{2r}} C}.$$

# Proof of Main Result

In [CY23], Corbett and Young substituted $z = -\frac{d}{c} + \frac{i}{c} \in \mathbb{H}$ into

$$S_{\chi_1,\chi_2}(a,c) = \frac{\tau(\overline{\chi_1})}{\pi i}\phi_{\chi_1,\chi_2}(\gamma, z).$$

This yields $\gamma z = \frac{a}{c} + \frac{i}{c}$. Their reason for choosing this value of $z$ is because it maximizes

$$\min\{\Im \gamma z, \Im z\}.$$

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$$\min\{\Im\gamma z, \Im z\}.$$

This proved useful in obtaining upper bounds for $f_{\chi_1,\chi_2}$ in [CY23]. Thus, we follow a similar approach and show that it suffices to bound

$$T(r,c) := \sum_{a \pmod{c}} \left| f_{\chi_1,\chi_2}\left(\frac{a}{c} + \frac{i}{c}\right)\right|^{2r} \tag{11}$$

to obtain our main result.

# Proof of Main Result

First, we show that

**Lemma**

$$\sum_{\substack{a \pmod{c} \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^{2r} \leq \frac{2^r q_1^r}{\pi^{2r}} T(r,c).$$

**Proof.**

We use the inequality

$$|x+y|^{2r} \leq 2^{r-1}(|x|^{2r} + |y|^{2r}),$$

substitute $z = \frac{-d}{c} + \frac{i}{c}$ into the definition of $S_{\chi_1,\chi_2}(a,c)$, use that $|\tau(\overline{\chi_1})| = \sqrt{q_1}$, and then reindex the sum. □

**Lemma**

*We have*

$$T(r,c) = c \sum_{\substack{m,n\geq 2 \\ m\equiv n \bmod c}} \exp\left(\frac{-2\pi(m+n)}{c}\right) S(m)\overline{S(n)},$$

*where*

$$S(n) := \sum_{t_1+\cdots+t_r=n} \lambda_{\chi_1,\chi_2}(t_1)\cdots\lambda_{\chi_1,\chi_2}(t_r) \tag{12}$$

*and*

$$\lambda_{\chi_1,\chi_2}(n) := \sum_{ab=n} \frac{\chi_1(a)}{a}\chi_2(b).$$

# Proof of Main Result

**Proof.**

The proof of this involves the expansion

$$\left| f_{\chi_1,\chi_2}\left(\frac{a}{c}+\frac{i}{c}\right)\right|^{2r} = f_{\chi_1,\chi_2}\left(\frac{a}{c}+\frac{i}{c}\right)^r \overline{f_{\chi_1,\chi_2}\left(\frac{a}{c}+\frac{i}{c}\right)}^r,$$

the orthogonality relations for additive characters

$$\frac{1}{q}\sum_{k=1}^{q} e\left(\frac{k(n-a)}{q}\right) = \begin{cases} 1, & n \equiv a \bmod q \\ 0, & \text{otherwise,} \end{cases}$$

and rewriting

$$\sum_{\ell_1 k_1 + \cdots + \ell_r k_r = n} \frac{\chi_1(\ell_1 \cdots \ell_r)\chi_2(k_1 \cdots k_r)}{\ell_1 \cdots \ell_r}$$

as $S(n)$. □

# Proof of Main Result

**Lemma**

$$S(n) \ll_r n^{r-1} \log^r n$$

**Proof.**

We use that $|\lambda_{\chi_1,\chi_2}(n)| \leq d(n)$, where $d(n) := \sum_{ab=n} 1$ is the divisor function, use the bounds $\sum_{t_1+t_2=n} d(t_1)d(t_2) \ll n \log^2 n$ and $\sum_{t_1=1}^{n-r+1} d(t_1) \ll n \log n$, and then induct on $r$. □

# Proof of Main Result

## Lemma

*For integers $k, m \geq 0$, we have*

$$\sum_{n\geq 2} \exp\left(\frac{-4\pi n}{c}\right) n^k \log^m n \ll_{k,m} c^{k+1} \log^m c.$$

## Proof.

This follows by bounding

$$\sum_{n=2}^{M} \exp\left(\frac{-4\pi n}{c}\right) n^k \log^m n, \qquad \sum_{n\geq M+1} \exp\left(\frac{-4\pi n}{c}\right) n^k \log^m n$$

where $M \in \mathbb{N}$, $M > \frac{k+m}{4\pi}c$, and $M \ll_{k,m} c$. □

# Proof of Main Result

**Lemma**

$$T(r,c) \ll_r c^{2r} \log^{2r} c$$

**Proof.**

This follows by applying our formula for $T(r,c)$, our bound for $S(n)$, the bound

$$(n+cj)^{r-1}\log^r(n+cj) \ll_r n^{r-1}\log^r n + (cj)^{r-1}\log^r c + (cj)^{r-1}\log^r j,$$

and the previous lemma. □

# Proof of Main Result

Combining this result with our first lemma

$$\sum_{\substack{a \pmod{c} \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^{2r} \leq \frac{2^r q_1^r}{\pi^{2r}} T(r,c),$$

we obtain our main result

**Theorem**

$$\sum_{\substack{a \pmod{c} \\ (a,c)=1}} |S_{\chi_1,\chi_2}(a,c)|^{2r} \ll_r q_1^r c^{2r} \log^{2r} c.$$

# Further Work

- Asymptotic formula for higher moments
- Improve the bound $S(n) \ll_r n^{r-1} \log^r n$
- Asymptotic formula for $S(n)$

# References


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