Visualizing and Understanding Tropical Geometry

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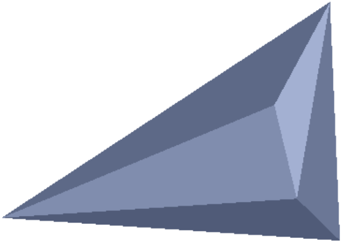
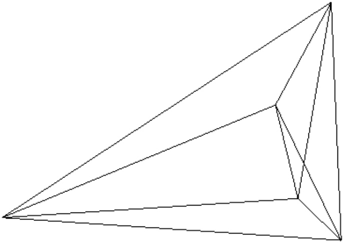
Geomview

- Computer Navigation
- XCode, XQuartz

Definitions

Definition

Real n-variate t-nomial:
$f(x) = \sum_{i=1}^{t} c_i x^{a_i}, \quad c_i \neq 0,\ a_i \in \mathbb{Z}^n$

Support: $Supp(f) = \{a_i \mid c_i \neq 0\}$

Definition

Positive Zero Set: $Z_+(f) = \{x \in \mathbb{R}_+^n \mid f(x) = 0\}$

Definitions

Definition

Circuit Case:

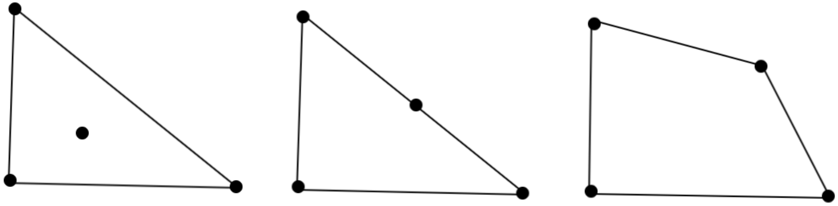
$\{a_1, \ldots, a_{n+2}\} \subset \mathbb{R}^n$ is a circuit iff the rank of the matrix

$$\begin{bmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{n+2} \end{bmatrix}$$

is $n+1$.

Definition

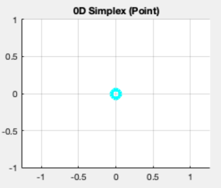
Newton Polytope: convex hull of the exponent vectors of the monomial terms of the polynomial

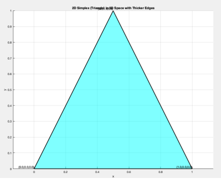


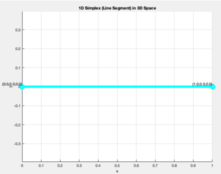
Simplex

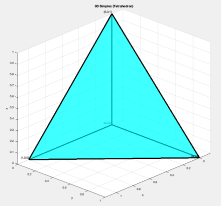
Definition

Simplex: Convex hull of $k+1$ affinely independent points in $\mathbb{R}^k$, where no subset of the points lie on the same hyperplane.









Plotting Surfaces

$$f(x, y, z) = 1y^{59}z^{47} + 3z^{79} + y^{181} + 1x^{233} - 10x^{81}y^{81}z^{81} = 0$$

- Small errors in x,y,z can lead to large errors
- Visual plots might not reflect the true surface
- For each (x,y), solve f(x,y,z) = 0 for z

Topological Information

Topological Invrients:

- Connected Components
- Holes
- Robustness

Why Topological Information about Zero Sets is Useful

- **Robotics:**
  - Path planning
  - Zero set describes constraints and obstacles in the space
- **Computational Biology:**
  - Zero sets can represent reaction pathways in biochemical processes
  - Shape and connectivity of zero sets can provide information into the stability and arrangement of molecular structures

Trop$_+$(f)

Definition

Trop$_+$(f): The set of $u^* \in \mathbb{R}^n$: $max|c_i e^{a_i u}|$ is attained at some pair of indices j and j' with $c_j c_j' < 0$

f(x,y)= -1+x+y

Trop(f)= $max\{-1, logx, logy\}$

$c_1 = -1$ $c_2 = 1$ $c_1 c_2 < 0$

$c_1 = -1$ $c_3 = 1$ $c_1 c_3 < 0$

$c_2 = -1$ $c_3 = 1$ $c_2 c_3 > 0$

Intersection: $-1 = \log x$

$-1 = \log y$

$u^* = (.1, .1)$

Rays: Extend in max directions

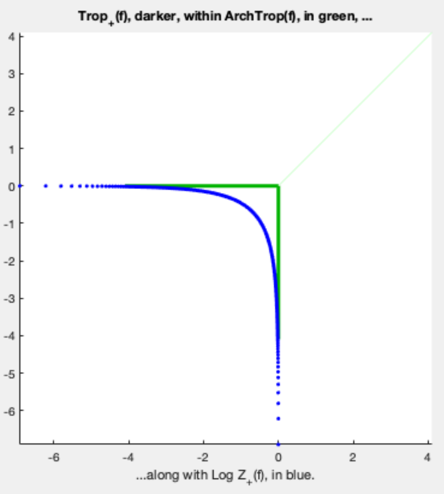
$a_2 - a_1$ & $a_3 - a_2$

$(1, 0)$ & $(0, 1)$

Approximating $\mathrm{LogZ}_+(f)$

f(x,y) = -1+x+y

- $\mathrm{Trop}_+(f)$ can be used to approximate $\mathrm{LogZ}_+(f)$
- $\mathrm{Trop}_+(f)$ and $\mathrm{LogZ}_+(f)$ are ambiently isotopic



Distance at most log(2)

Definition

Amoeba: The amoeba of a complex polynomial $f \in \mathbb{C}[x_1^{\pm}, ..., x_n^{\pm}]$ is the image of the zero set of f under the logarithmic map.

Theorem

For any $f \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ with exactly $t \geq 2$ monomial terms and $Newt(f)$ of dimension k, we have:

(a) $Amoeba(f) \subseteq ArchTrop(f)_{\log(t-1)}$

(b) *Let $\varphi(x) := 1 + x_1 + \cdots + x_{t-1}$:*

- *The amoeba $Amoeba(\varphi)$ contains a point at distance $\log(t-1)$ from $ArchTrop(\varphi)$.*
- *$ArchTrop(\varphi)$ contains points approaching a distance $\log(t-k)$ from $Amoeba(\varphi)$.*

Definition

b-vector: If Supp(f) forms a non-degenerate circuit, b-vector is defined by the coefficients b_i: $\sum_{i=1}^{n+2} b_i a_i = 0$

Definition

Sign-compatible: $sign(b_1 c_1) = ... = sign(b_{n+2} c_{n+2})$

Lemma

If Supp(f) can form a non-degenerate circuit, but f is sign incompatible, then $Z_+(f) = \emptyset$ if and only if all the coefficients of f have the same sign.

Isotopy and Distance Bounds

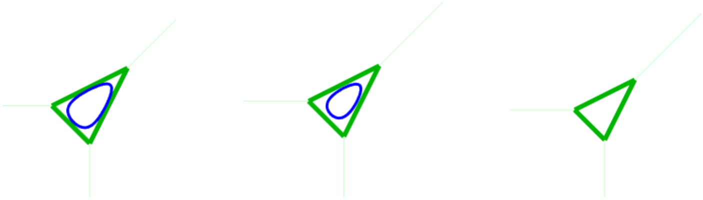
Theorem

Let $B := \sum_{i=1}^{n+2} b_i \log |b_i|$*. Then* $Z_+(f)$ *and* $Trop_+(f)$ *are ambiently isotopic in* $\mathbb{R}^n$*, and every point of* $LogZ_+(f)$ *is within distance log(n+1) of some point* $Trop_+(f)$ *if the conditions are met:*

1. **f is sign incompatible**
2. **f is sign compatible and**
   (a) $sign(B) \sum_{i=1}^{n+2} b_i \log |c_i| > 0$, *or*
   (b) $sign(B) \sum_{i=1}^{n+2} b_i \log |c_i| < -|B|$.

$\text{LogZ}_+(f)$ vs. $\text{Trop}_+(f)$

- Do not necessarily have the same isotopy type
- Consider $f(x, y) = 1 - cxy + x^3 + y^3$
- For most values of c, the isotopy types align



$c = 6$ $c = 4$ $c = 3$

$\text{Trop}_+(f)$ when $c \leq 3$ fails to capture the correct isotopy type

Newton Polytope: convex hull of the exponents of the terms of the polynomial

Toric Variety: Type of Algebraic Variety associated with torus actions

- Let $T = (\mathbb{C}^*)^n$ be an algebraic torus.
- A toric variety X has an algebraic torus T acting on it in a way that aligns with its geometric and algebraic structure

Moment Map: Maps points in the toric variety to a polytope

Constructing the Moment Map

- Moment map translates contributions of each monomial to a geometric representation
- Points generated are plottedto visualize the tropical variety

L1 Moment Map:

Sums of the absolute value of the monomials: $\lambda_i = \frac{a_i x^i}{\sum_j |a_j x^j|}$

L2 Moment Map:

Sums of the squares of the monomials: $\lambda_i = \frac{(a_i x^i)^2}{\sum_j (a_j x^j)^2}$

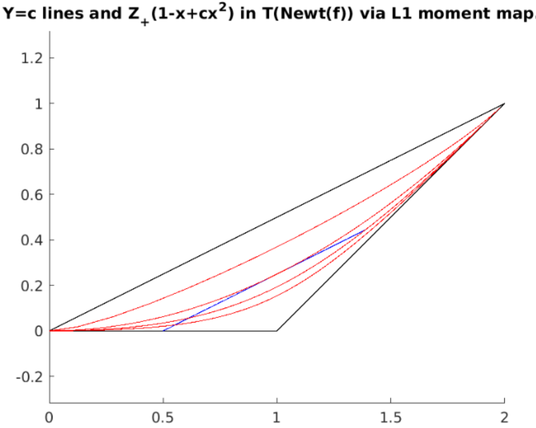
L3 Moment Map:

Sums of the exponents: $\lambda_i = \frac{e^{a_i x^i}}{\sum_j e^{a_i x^i}}$

Z_+ embedded in a curve inside a toric variety

f(x)= $1 - x_1 + cx_1^2$

- Newton polytope
- Blue: Moment map
- Red: Plotted for different values of c



Isotopy Type

Theorem

For any honest n-variate circuit polynomial f, the zero set $Z_+(f)$ is ambiently isotopic in $\mathbb{R}_+^n$ to $\mathbb{R}_+^{n-1}$ or to a zero set of the form $Z_\mathbb{R}(z_1^2 + z_2^2 + ... + z_l^2 - (z_{l+1}^2 + ... + z_m^2) + \epsilon)$, where $l \leq m \leq n$ and $\epsilon \in \{0, -1, 1\}$

Examples

Ambiently Isotopic to $\mathbb{R}_+^{n-1}$:

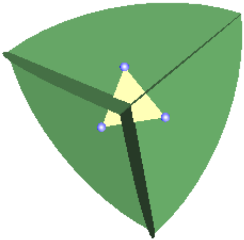
- f(x,y)=x-y
  - Straight line in xy plane. In $\mathbb{R}_+^{n}$, it represents a 1D subset which can be seen as $\mathbb{R}_+^{n-1}$

Isotopic to a quadratic form

- $x_1^2 + x_2^2 - x_3^2 = 1$
- $x_1^2 + x_2^2 - x_3^2 = 0$
- $x_1^2 + x_2^2 - x_3^2 = -1$

3-Variate 4-nomial

$$f(x, y, z) = x^{-1}yz + xyz^{-1} + xy^{-1}z + x^{-1}y^{-1}z^{-1}$$



3-Variate 5-nomial

$$f(x,y,z) = x^{-1}yz + xyz^{-1} + xy^{-1}z + x^{-1}y^{-1}z^{-1}$$