QUANTUM DYNAMICAL BOUNDS FOR QUASI-PERIODIC OPERATORS WITH LIOUVILLE FREQUENCIES

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ABSTRACT. We establish quantum dynamical upper bounds for quasi-periodic Schrödinger operators with Liouville frequencies. Our approach combines semi-algebraic discrepancy estimates for the Kronecker sequence $\{n\alpha\}$ with quantitative Green's function estimates adapted to the Liouville setting.

1. Introduction

In this paper, we study the quantum dynamics of the one-dimensional discrete Schrödinger operators. It is well-known that the solution of the time-dependent Schrödinger equation $i\partial_t\psi = H\psi$ is given by $\psi(t) = e^{-itH}\psi(0)$. For simplicity, we assume that the initial conditions $\psi(0) = \phi$ has compact support. The position operator is defined as

$$(X\psi)(n) = n\psi(n).$$

To describe the evolution of $\psi(t)$, we focus on the time-averaged p^{th} moment of the position operator defined via

$$\langle |X_H|_\phi^p\rangle(T) := \frac{2}{T}\int_0^\infty e^{-2\tau/T}\langle\psi(t), |X|^p\psi(t)\rangle d\tau.$$

Thus the growth of $T \mapsto \langle |X_H|_\phi^p\rangle(T)$ reflects the speed of which the particles spread out.

In particular, we consider the quasi-periodic Schrödinger operators $H = H_\theta$ on $\ell^2(\mathbb{Z})$,

$$(H_\theta\psi)_n = \psi_{n+1} + \psi_{n-1} + \lambda V(\theta + n\alpha)\psi_n,$$

where $V \in C^\omega(\mathbb{T}, \mathbb{R})$ is the potential, λ is the coupling constant, $\theta \in \mathbb{T}$ the phase, and $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is the frequency. The quantum dynamical behavior of H_θ depends heavily on the arithmetic property of the frequency α [Las96, GSB99, JZ22, LP22]. We say α is Diophantine if $\|n\alpha\|_{\mathbb{T}} \geqslant \eta|n|^{-\gamma}$ for some $\eta > 0, \gamma \geqslant 1$ where $\|\cdot\|_{\mathbb{T}} := \operatorname{dist}(\cdot, \mathbb{Z})$. We say α is Liouville if it is not Diophantine.

It has been shown that in the regime of large coupling λ and Diophantine frequency α, the quantity $\langle |X_{H_\theta}|_\phi^p\rangle(T)$ remains finite for almost every θ, and

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we achieve the so called dynamical localization, see [Bou05a, GYZ23, BJ00, JKL20]. However, dynamical localization fails to hold for generic $\theta \in \mathbb{T}$ [JS94, JL24, JLM24]. As a result, studying the growth of $\langle |X_{H_\theta}|^p_\phi \rangle(T)$ uniformly in θ becomes significantly more challenging and interesting.

For Diophantine frequency α, the growth behavior of $\langle |X_{H_\theta}|^p_\phi \rangle(T)$ has been investigated through various methods. For instance, Damanik–Tcheremchantsev [DT07, DT08] proved an upper bound of T^ε for trigonometric polynomial potentials using the transfer matrix method. Han–Jitomirskaya [HJ19] established a power-logarithmic bound $(\log T)^C$ for a family of ergodic potentials. Powell–Jitomirskaya [JP22] further proved a power-logarithmic bound $(\log T)^C$ by combining the large deviation theorem for transfer matrices with techniques from Damanik–Tcheremchantsev [DT07, DT08]. Additionally, Jitomirskaya–Liu [JL21] introduced a new approach based on Green's function estimates at suitable scales, which works for long-range operators. For Diophantine frequencies, they established an upper bound of the form T^ε. More recently, Shamis–Sodin [SS23] developed a method applicable to operators on $\mathbb{Z}^d (d \geqslant 1)$, using quantitative Green's function estimates from [Liu22].

In [Liu23], Liu established quantum dynamical upper bounds for long-range operators on $\mathbb{Z}^d$ based on the sublinear bounds for the semi-algebraic discrepancy:

$$\#\{\, |n| \leqslant N : \theta + n\alpha \bmod \mathbb{Z} \in \Theta \,\} \leqslant N^{1-\delta},$$

where Θ is a semi-algebraic set with suitable complexity. In [LPT⁺25, LPW24], the authors further developed this approach by applying tools from analytic number theory to analyze the discrepancy of semi-algebraic sets in the setting of quasi-periodic operators with multi-frequency shift and skew-shift potentials.

All known results establishing power-logarithmic upper bounds of the form $(\log T)^C$ crucially rely on the Diophantine condition for α. This naturally leads to the question: what is the growth behavior of $\langle |X_{H_\theta}|^p_\phi \rangle(T)$ when α is Liouville? For Liouville frequencies, sublinear discrepancy bounds fail, rendering the methods of [Liu23, LPT⁺25, LPW24] inapplicable.

A key observation in this work is that the existence of just one suitably chosen box of the Green's function is already sufficient to obtain a power-logarithmic bound of the form $(\log T)^C$. In contrast, the earlier result by Jitomirskaya–Liu [JL21] achieved only a T^ε bound under a similar one-box assumption.

In this paper, we first prove a general criterion (Theorem 3.1) for deriving quantum dynamical upper bounds. Notably, this criterion applies even to long-range operators without underlying dynamical systems. The criterion is based solely on the existence of a single suitably chosen box of the Green's function.

We then turn to discrepancy estimates for shift dynamics $\{n\alpha\}$ on semi-algebraic sets in the case of Liouville frequencies. As a consequence, we verify the existence of a box of the desired Green's function, extending the sublinear bounds known in the Diophantine case to a weaker, yet still effective, setting.

As applications to quasi-periodic operators, we obtain the following results:

Theorem 1.1. *Let $V \in C^\omega(\mathbb{T},\mathbb{R})$ be non-constant and $\phi \in \ell^2(\mathbb{Z})$ be compactly supported. Let $\eta > 0, \gamma \geqslant 1$. Suppose that $\alpha \in \mathbb{R}$ satisfies*

$$\|n\alpha\|_{\mathbb{T}} \geqslant \eta |n|^{-\gamma} \quad \text{for all } n \in \mathbb{Z}\setminus\{0\}.$$

Then there exist constants $C_0 > 0$ and $\lambda_0(V) > 0$ such that if $\lambda > \lambda_0$, the following holds. For any $p > 0, \varepsilon > 0$, there exists $T_1(\alpha, V, \phi, p, \varepsilon) > 0$ such that for $T \geqslant T_1$,

$$\sup_{\theta\in\mathbb{T}} \langle |X_{H_\theta}|_\phi^p\rangle(T) \leqslant (\log T)^{pC_0\gamma+\varepsilon}.$$

Theorem 1.2. *Let $V \in C^\omega(\mathbb{T},\mathbb{R})$ be non-constant and $\phi \in \ell^2(\mathbb{Z})$ be compactly supported. Let $\eta > 0, \kappa > 0, \gamma > 1$. Suppose that $\alpha \in \mathbb{R}$ satisfies*

$$\|n\alpha\|_{\mathbb{T}} \geqslant \eta e^{-\kappa(\log|n|)^\gamma} \quad \text{for all } n \in \mathbb{Z}\setminus\{0\}.$$

Then there exist constants $C_0 > 0$ and $\lambda_0(V) > 0$ such that if $\lambda > \lambda_0$, the following holds. For any $p > 0, \varepsilon > 0$, there exists $T_2(\alpha, V, \phi, p, \varepsilon) > 0$ such that for $T \geqslant T_2$,

$$\sup_{\theta\in\mathbb{T}} \langle |X_{H_\theta}|_\phi^p\rangle(T) \leqslant \exp\left(p\kappa(C_0+\varepsilon)^\gamma(\log\log T)^\gamma\right).$$

Theorem 1.3. *Let $V \in C^\omega(\mathbb{T},\mathbb{R})$ be non-constant and $\phi \in \ell^2(\mathbb{Z})$ be compactly supported. Let $\eta > 0, \kappa > 0, 0 < \gamma < \frac{1}{C_0}$. Suppose that $\alpha \in \mathbb{R}$ satisfies*

$$\|n\alpha\|_{\mathbb{T}} \geqslant \eta e^{-\kappa|n|^\gamma} \quad \text{for all } n \in \mathbb{Z}\setminus\{0\}.$$

Then there exists constant $\lambda_0(V) > 0$ such that if $\lambda > \lambda_0$, the following holds. For any $p > 0, \varepsilon > 0$, there exists $T_3(\alpha, V, \phi, p, \varepsilon) > 0$ such that for $T \geqslant T_3$,

$$\sup_{\theta\in\mathbb{T}} \langle |X_{H_\theta}|_\phi^p\rangle(T) \leqslant \exp\left(p(\log T)^{C_0\gamma+\varepsilon}\right).$$

Remark 1.4. In Theorem 1.1, 1.2, and 1.3, the constant $C_0 = 5C$, where $C \geqslant 1$ is the constant $C(d)$ from Lemma 2.5 when $d = 1$.

Theorem 1.1 is not new; it was previously established in [JL21, Liu23, JP22, LPT+25]. In fact, those works provide stronger versions with explicit estimates on the constants, which our approach does not yield. However, we include the result here to demonstrate the flexibility and effectiveness of our method. Theorems 1.2 and 1.3 are new. In the same setting, we note that

Damanik–Tcheremchantsev [DT07, DT08] proved an upper bound T^ε. Theorems 1.2 and 1.3 thus provide quantitative estimates for this T^ε in the case of Liouville frequencies.

2. Preliminaries

2.1. **Transfer matrix and Lyapunov exponent.** Denote by $C_h^\omega(\mathbb{T},\mathbb{R})$ the space of real-valued bounded analytic functions on the strip $\{\theta : |\Im\theta| < h\}$. For any $V \in C_h^\omega(\mathbb{T},\mathbb{R})$, define

$$\|V\|_h = \sup_{|\Im\theta|<h} |V(\theta)|.$$

Let $C^\omega(\mathbb{T},\mathbb{R}) := \bigcup_{h>0} C_h^\omega(\mathbb{T},\mathbb{R})$.

Denote

$$S_E^{\lambda V}(\theta) := \begin{pmatrix} E-\lambda V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

For any finite interval $\Lambda = [x_1, x_2] \subseteq \mathbb{Z}$ with $x_1 < x_2$, define the transfer matrix from x_1 to x_2 as

$$M_\Lambda(\theta) := \prod_{k=x_2-1}^{x_1} S_E^{\lambda V}(\theta + k\alpha).$$

Let $H_\Lambda(\theta) := R_\Lambda H_\theta R_\Lambda$, where R_Λ is the projection onto Λ. In particular, for $\Lambda = [0, N-1]$, denote $H_N(\theta) := H_{[0,N-1]}(\theta)$ and $M_N(\theta) := M_{[0,N-1]}(\theta)$.

It is straightforward to verify that

$$M_{[x_1,x_2]}(\theta) = M_{x_2-x_1+1}(\theta + x_1\alpha).$$

For $N \geqslant 1$, it is well-known (see [Bou05a]) that

$$M_N(\theta) = \begin{pmatrix} \det(H_N(\theta)-E) & -\det(H_{N-1}(\theta+\alpha)-E) \\ \det(H_{N-1}(\theta)-E) & -\det(H_{N-2}(\theta+\alpha)-E) \end{pmatrix}, \tag{1}$$

with the convention $\det(H_0(\theta)-E) := 1$ and $\det(H_{-1}(\theta)-E) := -1$.

Define the finite-scale Lyapunov exponent as

$$L_N(E) := \frac{1}{N}\int_{\mathbb{T}} \log\|M_N(\theta)\|\, d\theta,$$

and the Lyapunov exponent as

$$L(E) := \lim_{N\to\infty} L_N(E).$$

We recall the following result due to Furman:

Lemma 2.1 ([Fur97])**.** *Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $V \in C^\omega(\mathbb{T},\mathbb{R})$. Then*

$$\lim_{|\Lambda|\to\infty} \sup_{\theta\in\mathbb{T}} \frac{1}{|\Lambda|} \log\|M_\Lambda(\theta)\| = L(E).$$

The following lemma provides a criterion for the positivity of the Lyapunov exponent.

Lemma 2.2 ([Bou05b]). *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Suppose $V \in C^\omega(\mathbb{T}, \mathbb{R})$ is non-constant. Then there exists $\lambda_0(V) > 0$ such that for all $\lambda > \lambda_0$,*

$$L(E) > \frac{1}{2} \log \lambda \quad \textit{for all } E \in \mathbb{R}.$$

Remark 2.3. According to the Thouless formula, Lemma 2.2 in fact yields that $L(z) > \frac{1}{2} \log \lambda$ for all $z \in \mathbb{C}$.

2.2. **Green's Function.** For $z \notin \sigma(H_\theta)$, the Green's function of H_θ at z is defined by

$$G(z, \theta) := (H_\theta - zI)^{-1},$$

and for a finite interval Λ, define

$$G_\Lambda(z, \theta) := (H_\Lambda(\theta) - zI)^{-1}.$$

By Cramer's rule, for $\Lambda = [x_1, x_2]$ with $x_1 < x_2$ and $x_1 \leqslant m \leqslant n \leqslant x_2$, we have

$$G_\Lambda(z, \theta)(m, n) = \frac{\det(H_{m-x_1}(\theta + x_1\alpha) - z) \cdot \det(H_{x_2-n}(\theta + n\alpha) - z)}{\det(H_\Lambda(\theta) - z)}.$$

Using (1), it follows that

$$|G_\Lambda(z, \theta)(m, n)| \leqslant \frac{\|M_{m-x_1}(\theta + x_1\alpha)\| \cdot \|M_{x_2-n}(\theta + n\alpha)\|}{|\det(H_\Lambda(\theta) - z)|}. \tag{2}$$

2.3. **Semi-algebraic sets.**

Definition 2.4 ([Bou05a]). We say $\mathcal{S} \subseteq \mathbb{R}^d$ is a semi-algebraic set if it is a finite union of sets defined by a finite number of polynomial inequalities. More precisely, let $\{P_1, P_2, \cdots, P_s\}$ be a family of real polynomials to the variables $x = (x_1, x_2, \cdots, x_d)$ with $\deg(P_i) \leqslant q$ for $i = 1, 2, \cdots, s$. A (closed) semi-algebraic set $\mathcal{S}$ is given by the expression

$$\mathcal{S} = \bigcup_j \bigcap_{\ell \in \mathcal{L}_j} \{x \in \mathbb{R}^d : P_\ell(x) \varsigma_{j\ell} 0\}, \tag{3}$$

where $\mathcal{L}_j \subseteq \{1, 2, \cdots, s\}$ and $\varsigma_{j\ell} \in \{\geqslant, \leqslant, =\}$. Then we say that the degree of $\mathcal{S}$, denoted by $\deg(\mathcal{S})$, is at most sq. In fact, $\deg(\mathcal{S})$ means the smallest sq overall representation as in (3).

The following lemma has been stated in [Bou05a], where the author mentioned that it follows from the Yomdin–Gromov triangulation theorem [Gro87, Yom87]. For the history and the complete proof of the Yomdin–Gromov triangulation theorem, see [BN19].

Lemma 2.5 ([Bou05a]). *Let $\mathcal{S} \subseteq [0,1]^d$ be a semi-algebraic set of degree B. Let $\epsilon > 0$ be a small number and $\mathrm{Leb}(\mathcal{S}) \leqslant \epsilon^d$. Then $\mathcal{S}$ can be covered by a family of ϵ-balls with total number less than $B^{C(d)}\epsilon^{1-d}$.*

2.4. **Discrepancy.**

Definition 2.6 ([DT97]). Let $\{x_n\}_{n=1}^N$ be a sequence in $[0,1]^d$. The discrepancy of x_n is defined as

$$D_N(x_n) = \sup_{I \in R} \left| \frac{\#\{1 \leqslant n \leqslant N : x_n \in I\}}{N} - \mathrm{Leb}(I) \right|,$$

where R denotes the family of all axis-aligned rectangles in $[0,1]^d$.

In particular, for $x_n = \theta + n\alpha \mod \mathbb{Z}$, we denote by $D_N(\alpha)$ the discrepancy for short. The Erdős–Turán–Koksma inequality provides an upper bound for discrepancy.

Theorem 2.7 ([DT97, Erdős–Turán–Koksma Inequality]). *Let $\{x_n\}_{n=1}^N$ be a sequence in $[0,1]^d$ and $M \in \mathbb{N}$ be arbitrary. Then*

$$D_N(x_n) \leqslant \left(\frac{3}{2}\right)^d \left(\frac{2}{M+1} + \sum_{0<|m|<M} \frac{1}{r(m)} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \langle m, x_n \rangle} \right| \right),$$

where $r(m) = \prod_{i=1}^d \max\{1, |m_i|\}$ and $|m| = \max_{1 \leqslant i \leqslant d} |m_i|$.

3. Criterion for quantum dynamics

In this section, we establish a criterion for quantum dynamical upper bounds based on Green's function estimates on the suitable scales. The following criterion works for the bounded self-adjoint long-range operators on $\ell^2(\mathbb{Z})$.

Theorem 3.1. *Let $\{V_n\} \in \ell^\infty(\mathbb{Z})$ be a real sequence, and consider the long-range operator*

$$(H\psi)_n = \sum_{m=-\infty}^{\infty} A_m \psi_{n-m} + V_n \psi_n,$$

where $\overline{A_m} = A_{-m}$ and $|A_m| \leqslant C_1 e^{-c_1|m|}$ for all $m \in \mathbb{Z}$. Assume that $\sigma(H) \subseteq [-K+1, K-1]$ for some $K \geqslant 3$.

Let $\phi \in \ell^2(\mathbb{Z})$ with $\mathrm{supp}\,\phi \subseteq [-M, M]$. Suppose that for every $|N| \geqslant N_0$, there exists an interval $I \subseteq [-\frac{|N|}{2}, -\frac{|N|}{4}] \cap \mathbb{Z}$ or $I \subseteq [\frac{|N|}{4}, \frac{|N|}{2}] \cap \mathbb{Z}$ satisfying the following conditions:

(1) There exists a monotone increasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|I| \geqslant \Psi(|N|) \geqslant (\log |N|)^{C_2}, \tag{4}$$

for some constant $C_2 > 1$.

(2) There exists $0 < c_2 \leqslant c_1$ such that for any $z = E + i\epsilon$ with $|E| \leqslant K$ and $0 < \epsilon \leqslant \epsilon_0$,

$$|G_I(z)(m,n)| < e^{-c_2|I|}, \quad \text{for all } |m-n| > \frac{|I|}{20}. \tag{5}$$

Then for any $p > 0$, there exists $T_0 = T_0(M, K, C_1, C_2, c_1, c_2, \epsilon_0, p)$ such that for all $T \geqslant T_0$,

$$\langle |X_H|_\phi^p \rangle(T) \leqslant \left[\Gamma\left(\frac{80}{c_2} \log T \right) \right]^p,$$

where $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is the inverse function of Ψ.

Proof. Let $z = E + \frac{i}{T}$ where $|E| \leqslant K$ and $T \geqslant \frac{1}{\epsilon_0}$. Recall the following lemma essentially proved by Jitomirskaya–Liu [JL21].

Lemma 3.2 ([JL21, Lemma 2.3])**.** *Assume for some interval I with $I \subseteq [-\frac{|N|}{2}, -\frac{|N|}{4}]$ or $I \subseteq [\frac{|N|}{4}, \frac{|N|}{2}]$, (5) holds. Then for any $|j| \leqslant M$,*

$$|G(z)(j,N)| \lesssim_{C_1,c_1} T^4 e^{-\frac{c_2}{20}|I|}.$$

By Lemma 3.2 and (4), for any $|j| \leqslant M$ and $|n| \geqslant N_0$,

$$|G(z)(j,n)| \lesssim_{C_1,c_1} T^4 e^{-c_3\Psi(|n|)}, \tag{6}$$

where $c_3 = \frac{c_2}{20}$. Denote

$$a(j,n,T) = \frac{2}{T}\int_0^\infty e^{-2\tau/T} |\langle \delta_n, e^{-iTH}\delta_j\rangle|^2 d\tau.$$

According to Parseval's identity (see [DF22]), one has

$$a(j,n,T) = \frac{1}{\pi T}\int_{\mathbb{R}} |G(z)(j,n)|^2 dE.$$

Since $\sum_{n\in\mathbb{Z}} |\langle \delta_n, e^{-iTH}\delta_j\rangle|^2 = 1$, one has

$$\sum_{n\in\mathbb{Z}} a(j,n,T) = 1 \quad \text{for any } |j| \leqslant M. \tag{7}$$

By (7), for any $R \geqslant 1$,

$$\begin{aligned} \langle |X_H|_\phi^p \rangle(T) &\leqslant \left(\sum_{|n|\leqslant R} + \sum_{|n|\geqslant R} \right) \sum_{|j|\leqslant M} |n|^p a(j,n,T) \\ &\lesssim_M R^p + \sum_{|j|\leqslant M} \sum_{|n|\geqslant R} |n|^p a(j,n,T). \end{aligned} \tag{8}$$

By the Combes–Thomas estimate (see [Aiz94]), for sufficiently large $|n|$,

$$\begin{aligned} a(j,n,T) &\leqslant \frac{1}{\pi T}\bigg(\int_{|E|\geqslant K} + \int_{|E|\leqslant K}\bigg)|G(z)(j,n)|^2 dE \\ &\lesssim_K \frac{1}{T}e^{-c_4|n|} + \frac{1}{T}\int_{|E|\leqslant K}|G(z)(j,n)|^2 dE, \end{aligned} \tag{9}$$

where $c_4 > 0$ is a universal constant. By (6), we have

$$\frac{1}{T}\int_{|E|\leqslant K}|G(z)(j,n)|^2 dE \lesssim_{C_1,c_1,K} T^7 e^{-2c_3\Psi(|n|)}. \tag{10}$$

Combining (8), (9) and (10), we get

$$\begin{aligned} \langle |X_H|_\phi^p\rangle(T) &\lesssim_{M,K,C_1,c_1} R^p + \frac{1}{T}\sum_{|n|\geqslant R}|n|^p e^{-c_4|n|} + T^7\sum_{|n|\geqslant R}|n|^p e^{-2c_3\Psi(|n|)} \\ &\lesssim_{M,K,C_1,c_1} R^p + T^7\sum_{|n|\geqslant R}|n|^p e^{-2c_3\Psi(|n|)}. \end{aligned}$$

Choose R such that $\Psi(R) = \frac{3.9}{c_3}\log T$, that is $R = \Gamma(\frac{3.9}{c_3}\log T)$. Hence for sufficiently large $T \geqslant T_0(M,K,C_1,C_2,c_1,c_3,\epsilon_0,p)$,

$$\langle |X_H|_\phi^p\rangle(T) \leqslant \left[\Gamma\bigg(\frac{4}{c_3}\log T\bigg)\right]^p.$$

This finishes the proof. □

In particular, we are interested in the following $\Psi(\cdot)$.

Corollary 3.3. *Under the assumptions of Theorem 3.1, if* $\Psi(\cdot)$ *takes the form*

$$\Psi(N) = N^\delta$$

for some $\delta > 0$*, then for any* $\varepsilon > 0$*, there exists* $T_1 = T_1(T_0,\delta,\varepsilon) > 0$ *such that for all* $T \geqslant T_1$*,*

$$\langle |X_H|_\phi^p\rangle(T) \leqslant (\log T)^{\frac{p}{\delta}+\varepsilon}.$$

Corollary 3.4. *Under the assumptions of Theorem 3.1, if* $\Psi(\cdot)$ *takes the form*

$$\Psi(N) = \exp\Big(\delta(\log N)^\sigma\Big),$$

for some $\delta,\sigma > 0$*, then for any* $\varepsilon > 0$*, there exists* $T_2 = T_2(T_0,\delta,\sigma,\varepsilon) > 0$ *such that for all* $T \geqslant T_2$*,*

$$\langle |X_H|_\phi^p\rangle(T) \leqslant \exp\left[p\bigg(\frac{1+\varepsilon}{\delta}\log\log T\bigg)^{1/\sigma}\right].$$

Corollary 3.5. *Under the assumptions of Theorem 3.1, if* $\Psi(\cdot)$ *takes the form*

$$\Psi(N) = (\log N)^{1/\delta}$$

for some $0 < \delta < 1$*, then for any* $\varepsilon > 0$*, there exists* $T_3 = T_3(T_0, \delta, \varepsilon) > 0$ *such that for all* $T \geqslant T_3$*,*

$$\langle |X_H|_\phi^p \rangle(T) \leqslant \exp\left(p(\log T)^{\delta+\varepsilon} \right).$$

4. Discrepancy estimates

We first prove a few fundamental results regarding discrepancy estimates.

Theorem 4.1. *Let* $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be such that* $\Phi(t)/t$ *is monotone increasing on* $[\rho, \infty)$ *for some* $\rho > 0$*. Let* $\mu := \max_{t\in[1,\rho]} \Phi(t) < \infty$*. Let* $\alpha \in \mathbb{R}^d$ *satisfy*

$$\|\langle n, \alpha\rangle\|_{\mathbb{T}} > \frac{1}{\Phi(|n|)} \text{ for any } n \in \mathbb{Z}^d \setminus \{0\}. \tag{11}$$

Then for any $M \geqslant \rho$*,*

$$D_N(\alpha) \lesssim_{\rho,\mu,d} \frac{1}{M} + \frac{1}{N} + \frac{\Phi(M)\log(\Phi(M))(\log M)^d}{MN}.$$

Proof. Without loss of generality, we assume $\rho \geqslant 2$. By Theorem 2.7, for any $M \in \mathbb{N}$ we have

$$D_N(\alpha) \lesssim_d \frac{1}{M} + \frac{1}{N} \sum_{|m|=1}^{M} \frac{1}{r(m)} \left| \sum_{k=1}^{N} e^{2\pi i k\langle m,\alpha\rangle} \right|,$$

where $r(m) = \prod_{i=1}^d \max\{1, |m_i|\}$ and $|m| = \max_{1\leqslant i\leqslant d} |m_i|$. Since

$$\left| \sum_{k=1}^{N} e^{2\pi i k x} \right| \lesssim \min\{N, \|x\|_{\mathbb{T}}^{-1}\},$$

we have

$$\frac{1}{N} \sum_{|m|=1}^{\rho} \frac{1}{r(m)} \left| \sum_{k=1}^{N} e^{2\pi i k\langle m,\alpha\rangle} \right| \lesssim \frac{1}{N} \sum_{|m|=1}^{\rho} \frac{1}{r(m) \cdot \|\langle m,\alpha\rangle\|_{\mathbb{T}}} \lesssim_{\rho,\mu} \frac{1}{N},$$

and thus

$$D_N(\alpha) \lesssim_{\rho,\mu,d} \frac{1}{M} + \frac{1}{N} + \frac{1}{N} \sum_{|m|=\rho}^{M} \frac{1}{r(m) \cdot \|\langle m,\alpha\rangle\|_{\mathbb{T}}} =: \frac{1}{M} + \frac{1}{N} + \frac{S_0}{N}. \tag{12}$$

Let $r = (r_1, \cdots, r_d) \in \mathbb{N}^d$ such that

$$\frac{\log \rho}{\log 2} \leqslant r_i \leqslant \frac{\log M}{\log 2} \quad \text{for each } 1 \leqslant i \leqslant d.$$

Fix any r and denote

$$T_r := \{m \in \mathbb{Z}^d : 2^{r_i-1} \leqslant |m_i| \leqslant 2^{r_i} \text{ for } 1 \leqslant i \leqslant d\}.$$

Thus

$$\begin{aligned} S_0 &= \sum_r \sum_{m\in T_r} \frac{1}{r(m)\cdot \|\langle m,\alpha\rangle\|_{\mathbb{T}}} \\ &\leqslant \sum_r 2^{-\sum_{i=1}^d (r_i-1)} \sum_{m\in T_r} \frac{1}{\|\langle m,\alpha\rangle\|_{\mathbb{T}}}. \end{aligned} \tag{13}$$

Without loss of generality, we assume r satisfies $r_1 = |r|$. By (11), for any $m \in T_r$ we have the estimate

$$\|\langle m,\alpha\rangle\|_{\mathbb{T}} \geqslant \frac{1}{\Phi(|m|)} \geqslant \frac{1}{\Phi(2^{r_1})}.$$

Define $\Delta = \Phi(2^{r_1})$. We need the following result:

Lemma 4.2. *For all $l \leqslant \lfloor \Delta \rfloor$, there are at most 2^{d+1} points m in T_r satisfying the inequality*

$$l\Delta^{-1} \leqslant \|\langle m,\alpha\rangle\|_{\mathbb{T}} \leqslant (l+1)\Delta^{-1}.$$

Proof. Otherwise if there exist $(2^{d+1}+1)$ points, then either there will be (2^d+1) points satisfying

$$\|\langle m,\alpha\rangle\|_{\mathbb{T}} = \{\langle m,\alpha\rangle\},$$

or there will be (2^d+1) points satisfying

$$\|\langle m,\alpha\rangle\|_{\mathbb{T}} = 1 - \{\langle m,\alpha\rangle\}.$$

In either case, there exists a hyperoctant of $\mathbb{Z}^d$ such that at least two points m, m' among (2^d+1) points are located in this hyperoctant, which means

$$m_j m'_j \geqslant 0, \text{ for all } j = 1, \cdots, d.$$

It would follow that

$$0 < |m - m'| < 2^{r_1-1}$$

and

$$\begin{aligned} \|\langle m-m',\alpha\rangle\|_{\mathbb{T}} &\leqslant \{\langle m-m',\alpha\rangle\} \\ &= |\{\langle m,\alpha\rangle\} - \{\langle m',\alpha\rangle\}| \\ &= |\|\langle m,\alpha\rangle\|_{\mathbb{T}} - \|\langle m',\alpha\rangle\|_{\mathbb{T}}| \\ &\leqslant \Delta^{-1}. \end{aligned}$$

Using (11), we arrive at

$$\|\langle m-m',\alpha\rangle\|_{\mathbb{T}} \geqslant \frac{1}{\Phi(|m-m'|)} > \frac{1}{\Phi(2^{r_1})} = \Delta^{-1},$$

which is a contradiction. ☐

From Lemma 4.2, we see that

$$\sum_{m\in T_r}\frac{1}{\|\langle m,\alpha\rangle\|_{\mathbb{T}}}\lesssim_d \Delta\sum_{l=1}^{[\Delta]}\frac{1}{l}\lesssim_d \Delta\log(\Delta).$$

Substituting the above estimate into (13) then yields

$$S_0\lesssim_d \sum_{r_1=\log_2\rho}^{\log_2 M} 2^{-r_1}\Delta\log(\Delta)(\log M)^{d-1}.$$

By the assumption that $\Phi(t)/t$ is monotone increasing, we thus have

$$S_0\lesssim_d M^{-1}\Phi(M)\log(\Phi(M))(\log M)^d. \tag{14}$$

Substituting (14) into (12) gives

$$D_N(\alpha)\lesssim_{\rho,\mu,d}\frac{1}{M}+\frac{1}{N}+\frac{\Phi(M)\log(\Phi(M))(\log M)^d}{MN}.$$

This finishes the proof. ☐

In particular, we are interested in the following cases.

Corollary 4.3. *Let $\eta>0,\gamma\geqslant 1$. Let $\alpha\in\mathbb{R}^d$ satisfy*

$$\|\langle n,\alpha\rangle\|_{\mathbb{T}}\geqslant \eta|n|^{-\gamma}\quad \text{for all } n\in\mathbb{Z}^d\setminus\{0\}.$$

Then for sufficiently large N,

$$D_N(\alpha)\lesssim_{d,\gamma,\eta} N^{-1/\gamma}(\log N)^{d+1}.$$

Proof. Let $\Phi(t)=\eta^{-1}t^\gamma$. Then $\Phi(t)/t$ is monotone increasing on $[2,\infty)$. Apply Theorem 4.1 with $\Phi(M)=N$, that is,

$$M=(\eta N)^{1/\gamma},$$

we have

$$\begin{aligned}D_N(\alpha)&\lesssim_{d,\gamma,\eta}\frac{1}{M}+\frac{1}{N}+\frac{\log N(\log M)^d}{M}\\&\lesssim_{d,\gamma,\eta} N^{-1/\gamma}(\log N)^{d+1}.\end{aligned}$$

☐

Corollary 4.4. *Let $\eta>0,\kappa>0,\gamma>1$. Let $\alpha\in\mathbb{R}^d$ satisfy*

$$\|\langle n,\alpha\rangle\|_{\mathbb{T}}\geqslant \eta e^{-\kappa(\log|n|)^\gamma}\quad \text{for all } n\in\mathbb{Z}^d\setminus\{0\}.$$

Then for sufficiently large N,

$$D_N(\alpha)\lesssim_{d,\kappa,\gamma,\eta}(\log N)^{1+d/\gamma}\exp\left[-\left(\frac{1}{\kappa}\log N\right)^{1/\gamma}\right].$$

Proof. Let $\Phi(t) = \eta^{-1} e^{\kappa(\log t)^\gamma}$. Then there exists ρ (depending on κ, γ) such that $\Phi(t)/t$ is monotone increasing on $[\rho, \infty)$. Apply Theorem 4.1 with $\Phi(M) = N$, that is,

$$M = \exp\left[\left(\frac{1}{\kappa}\log(\eta N)\right)^{1/\gamma}\right],$$

we have

$$\begin{aligned} D_N(\alpha) &\lesssim_{\kappa,\gamma,d,\eta} \frac{1}{M} + \frac{1}{N} + \frac{\log N (\log M)^d}{M} \\ &\lesssim_{d,\kappa,\gamma,\eta} (\log N)^{1+d/\gamma} \exp\left[-\left(\frac{1}{\kappa}\log N\right)^{1/\gamma}\right]. \end{aligned}$$

□

Corollary 4.5. *Let $\eta > 0, \kappa > 0, 0 < \gamma < 1$. Let $\alpha \in \mathbb{R}^d$ satisfy*

$$\|\langle n, \alpha\rangle\|_{\mathbb{T}} \geqslant \eta e^{-\kappa|n|^\gamma} \quad \textit{for all } n \in \mathbb{Z}^d \setminus \{0\}.$$

Then for sufficiently large N,

$$D_N(\alpha) \lesssim_{d,\kappa,\gamma,\eta} (\log N)^{-1/\gamma}.$$

Proof. Let $\Phi(t) = \eta^{-1} e^{\kappa t^\gamma}$. Then there exists ρ (depending on κ, γ) such that $\Phi(t)/t$ is monotone increasing on $[\rho, \infty)$. Apply Theorem 4.1 with $\Phi(M) = \sqrt{N}$, that is,

$$M = \left(\frac{1}{\kappa}\log(\eta\sqrt{N})\right)^{1/\gamma},$$

we have

$$\begin{aligned} D_N(\alpha) &\lesssim_{d,\kappa,\gamma,\eta} \frac{1}{M} + \frac{1}{N} + \frac{\log N (\log M)^d}{M\sqrt{N}} \\ &\lesssim_{d,\kappa,\gamma,\eta} (\log N)^{-1/\gamma}. \end{aligned}$$

□

The following semi-algebraic discrepancy estimate is useful and will be applied repeatedly.

Theorem 4.6. *Suppose $D_N(x_n) \leqslant Y_N \xrightarrow{N\to\infty} 0$. Let $\mathcal{S} \subseteq [0,1]^d$ be a semi-algebraic set with degree B and $\mathrm{Leb}(\mathcal{S}) < Y_N$. Then*

$$\#\{x_n \in \mathcal{S}\} \leqslant 2B^{C(d)} N Y_N^{1/d}.$$

Proof. Let $\epsilon = Y_N^{1/d}$. Then

$$\mathrm{Leb}(\mathcal{S}) < Y_N = \epsilon^d.$$

By Lemma 2.5, the set $\mathcal{S}$ can be covered by at most $B^{C(d)}\epsilon^{1-d}$ balls of radius ϵ. Let D be one such ball. Then by the definition of discrepancy,

$$\#\{x_n \in D\} \leqslant N\,\mathrm{Leb}(D) + ND_N(\alpha) \leqslant 2NY_N.$$

Summing over all such balls, we obtain

$$\#\{x_n \in \mathcal{S}\} \leqslant 2B^{C(d)}NY_N^{1/d}.$$

This finishes the proof. □

Theorem 4.6 was proved for $Y_N = N^{-\varsigma}$ with $\varsigma > 0$ in [Liu22].

5. Large deviation theorem for Liouville frequencies

Since we are interested in quantum dynamics with Liouville frequencies, we will need the following large deviation theorem for transfer matrices, which holds in the Liouville setting.

Theorem 5.1 ([HZ22]). *Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $V \in C_h^\omega(\mathbb{T},\mathbb{R})$. There exist constants $\tilde{c}_1(V,h), \tilde{c}_2(V,h) \in (0,1)$ such that, if* [1]

$$0 \leqslant \beta(\alpha) < \tilde{c}_1 \inf_{E\in[a,b]} L(E),$$

there exists $N_1 = N_1(\alpha, \inf_{E\in[a,b]} L(E), V, h) > 0$ such that for any $N \geqslant N_1$, the following large deviation estimates hold uniformly in $E \in [a,b]$:

(1) If $0 < L(E) < 1$, then

$$\mathrm{Leb}\left\{\theta : \left|\frac{1}{N}\log\|M_N(\theta)\| - L_N(E)\right| > \frac{1}{100}L(E)\right\} < e^{-\tilde{c}_2 L(E)N}.$$

(2) If $L(E) \geqslant 1$, then

$$\mathrm{Leb}\left\{\theta : \left|\frac{1}{N}\log\|M_N(\theta)\| - L_N(E)\right| > \frac{1}{100}L(E)\right\} < e^{-\tilde{c}_2 L(E)^2 N}.$$

Remark 5.2. In fact, $\frac{1}{100}$ in Theorem 5.1 can be replaced by any $0 < \kappa < 1$. See Remark 1.2 in [HZ22].

The conclusion of Theorem 5.1 remains valid for complex energies $E + i\epsilon$, where $E \in [a,b]$ and $|\epsilon| \leqslant \epsilon_0$ for some sufficiently small $\epsilon_0 > 0$ because the proof of Theorem 5.1 only involves the subharmonicity of the Lyapunov exponent. More precisely, we have the following:

[1] Recall that $\beta(\alpha) = \limsup_{|k|\to\infty} -\frac{\log\|k\alpha\|_{\mathbb{T}}}{|k|}$.

Corollary 5.3. *Under the same assumptions as in Theorem 5.1, there exist* $N_1 = N_1(\alpha, V, a, b) > 0$ *and* $\epsilon_0 = \epsilon_0(\alpha, V, a, b) > 0$ *such that for any* $N \geqslant N_1$*, the following holds uniformly for* $z = E + i\epsilon$ *where* $E \in [a, b]$ *and* $|\epsilon| \leqslant \epsilon_0$*:*

$$\mathrm{Leb}\left\{\theta : \left|\frac{1}{N}\log\|M_N(\theta)\| - L_N(z)\right| > \frac{1}{100}L(z)\right\} < e^{-\tilde{c}_2 L(z)N}.$$

With Theorem 5.1 and Corollary 5.3 in hand, we deduce the following large deviation theorem for Green's function in the Liouville setting.

Theorem 5.4. *Let* $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ *and* $V \in C_h^\omega(\mathbb{T}, \mathbb{R})$*. There exist constants* $\tilde{c}_1(V, h), \tilde{c}_2(V, h) \in (0, 1)$ *such that, if*

$$0 \leqslant \beta(\alpha) < \tilde{c}_1 \inf_{E\in[a,b]} L(E),$$

there exist $N_2 = N_2(\alpha, V, a, b) > 0$ *and* $\epsilon_0 = \epsilon_0(\alpha, V, a, b) > 0$ *such that for any* $N \geqslant N_2$*, and* $z = E + i\epsilon$ *with* $E \in [a, b]$ *and* $|\epsilon| \leqslant \epsilon_0$*, the following holds. There exists a subset* $\Theta_N \subseteq \mathbb{T}$ *(depending on* z*) with*

$$\mathrm{Leb}(\Theta_N) \leqslant e^{-\tilde{c}_2 L(z)(2N+1)}, \quad \deg(\Theta_N) \lesssim_h L(z)^2 N^5,$$

such that for any $\theta \notin \Theta_N$*, one of the intervals*

$$\Lambda = [-N, N]; [-N, N-1]; [-N+1, N]; [-N+1, N-1]$$

will satisfy

$$|G_\Lambda(z)(m, n)| \leqslant e^{-\frac{1}{2}L(z)|m-n|}, \quad \text{for any } |m-n| > \frac{|\Lambda|}{20}.$$

Proof. By Corollary 5.3, for any $N \geqslant N_1(\alpha, V, a, b)$, there exists a subset $\Theta_N \subseteq \mathbb{T}$ with $\mathrm{Leb}(\Theta_N) \leqslant e^{-\tilde{c}_2 L(z)(2N+1)}$ such that for any $\theta \notin \Theta_N$,

$$\|M_{[-N,N]}(\theta)\| \geqslant e^{\frac{99}{100}(2N+1)L(z)}.$$

Then it follows from (1) that for one of the intervals

$$\Lambda = [-N, N]; [-N, N-1]; [-N+1, N]; [-N+1, N-1]$$

we have

$$|\det(H_\Lambda(\theta) - z)| \geqslant \frac{1}{4}e^{\frac{99}{100}(2N+1)L(z)}. \tag{15}$$

By Lemma 2.1 and the compactness argument, for any $\varepsilon > 0$, there exists $\bar{N}_1 = \bar{N}_1(\alpha, V, a, b, \varepsilon)$ such that for any $N \geqslant \bar{N}_1$, we have

$$\sup_{\theta\in\mathbb{T}} \|M_N(\theta)\| \leqslant e^{N(L(z)+\varepsilon)}. \tag{16}$$

It follows from (2),(15) and (16) that for any $\theta \notin \Theta_N$, we have

$$|G_\Lambda(\theta)(m, n)| \leqslant e^{\frac{1}{100}|\Lambda|L(z)} e^{|\Lambda|\varepsilon} e^{-L(z)|m-n|}, \text{ for any } m, n \in \Lambda.$$

In particular, for any $\theta \notin \Theta_N$ and $m, n \in \Lambda$ with $|m-n| \geqslant |\Lambda|/20$,

$$|G_\Lambda(\theta)(m,n)| \leqslant e^{-\frac{1}{2}L(z)|m-n|}.$$

In the following, we estimate the complexity. Consider the property

$$|G_\Lambda(\theta)(m,n)| \leqslant e^{\frac{1}{100}|\Lambda|L(z)} e^{|\Lambda|\varepsilon} e^{-L(z)|m-n|}, \text{ for any } m,n \in \Lambda. \tag{17}$$

Since $V \in C_h^\omega(\mathbb{T}, \mathbb{R})$, we write $V(\theta) = \sum_{k\in\mathbb{Z}} \widehat{V}_k e^{2\pi i k\theta}$. Consider its Fourier truncation $V_1(\theta) = \sum_{|k|\leqslant C'N} \widehat{V}_k e^{2\pi i k\theta}$ where $C' = \frac{100}{2\pi h} L(z)$. Then

$$\sup_{\theta\in\mathbb{T}} |V_1(\theta) - V(\theta)| \leqslant \|V\|_h e^{-2\pi h C' N} \leqslant \|V\|_h e^{-100L(z)N}.$$

By (1) and a telescoping argument, for any $1 \leqslant J \leqslant |\Lambda|$, we have

$$\begin{aligned}
&\sup_{\theta\in\mathbb{T}} |\det(H_J(\theta, V) - z) - \det(H_J(\theta, V_1) - z)| \\
&\qquad \leqslant \sup_{\theta\in\mathbb{T}} \left\| \prod_{j=1}^{J} S_z^V(\theta + j\alpha) - \prod_{j=1}^{J} S_z^{V_1}(\theta + j\alpha) \right\| \\
&\qquad \leqslant \sup_{\theta\in\mathbb{T}} |V_1(\theta) - V(\theta)| J e^{J(L(z)+\varepsilon)} \\
&\qquad \leqslant e^{-20|\Lambda|L(z)},
\end{aligned}$$

which implies for any $\theta \in \Theta_N$,

$$|G_\Lambda(\theta, V)(m,n) - G_\Lambda(\theta, V_1)(m,n)| \leqslant e^{-10|\Lambda|L(z)}.$$

Thus we may substitute V by V_1 in (17), that is,

$$e^{L(z)|m-n|} |\det(H_\Lambda(V_1, \theta) - E)_{m,n}| \leqslant \frac{1}{2} e^{\frac{1}{100}|\Lambda|L(z)} e^{|\Lambda|\varepsilon} |\det(H_\Lambda(V_1, \theta) - z)|, \tag{18}$$

where $M_{m,n}$ denotes the (m,n)-minor of matrix M. Using the Hilbert-Schmidt norm, we may consider the tighter inequality,

$$\begin{aligned}
\sum_{m,n\in\Lambda} & e^{2L(z)|m-n|} \Big(\det(H_\Lambda(V_1, \theta) - z)_{m,n} \Big)^2 \\
&\qquad \leqslant \frac{1}{4} e^{\frac{1}{50}|\Lambda|L(z)} e^{|\Lambda|\varepsilon} \Big(\det(H_\Lambda(V_1, \theta) - z) \Big)^2.
\end{aligned} \tag{19}$$

Clearly, (19) is of the form

$$P_1(\cos 2\pi\theta, \sin 2\pi\theta) \geqslant 0,$$

where P_1 is a polynomial with degree at most $C'^2 N^4$. One further truncates "cos" and "sin" by a Taylor polynomial of degree at most N, and get a tighter inequality of the form

$$P_2(\theta) \geqslant 0,$$

where the degree of P_2 is at most C'^2N^5. Denote $\Theta'_N = \{\theta \in [0,1] : P_2(\theta) < 0\}$. By the definition of the degree of semi-algebraic sets (see Definition 2.4), we have $\deg(\overline{\Theta'_N}) \leqslant C'^2N^5$.

For any $\theta \notin \Theta_N$, it is easy to check that $P_2(\theta) \geqslant 0$. Thus $\Theta'_N \subseteq \Theta_N$ and

$$\mathrm{Leb}(\overline{\Theta'_N}) \leqslant \mathrm{Leb}(\Theta_N) \leqslant e^{-\tilde{c}_2 L(z)(2N+1)}.$$

Finally one may replace Θ_N by $\overline{\Theta'_N}$. This finishes the proof. □

6. Proof of main results

We are now ready to prove our main results by combining the tools developed in Sections 3, 4 and 5.

6.1. Proof of Theorem 1.1.

Proof. Fix a small $\varepsilon > 0$ and take N sufficiently large. Let $C \geqslant 1$ be the constant $C(d)$ from Lemma 2.5 when $d = 1$. Define an interval I centered at 0 and of length $|I| = N^\delta$ with $\delta = \frac{1}{5\gamma C} - \varepsilon$. Set $I_j = I + j$ for $|j| \leqslant N$.

Recall that for any $\theta \in \mathbb{T}$, one has $H_{I_j}(\theta) = H_I(\theta + j\alpha)$. Then for any $z = E + \frac{i}{T}$,

$$G_{I_j}(z, \theta) = G_I(z, \theta + j\alpha). \tag{20}$$

By Lemma 2.2 and Theorem 5.4, there exists a subset $\Theta_I \subseteq \mathbb{T}$ such that $\mathrm{Leb}(\Theta_I) \leqslant e^{-\tilde{c}_2 L(z)|I|}$ and $\deg(\Theta_I) \lesssim_h L(z)^2|I|^5$, with the property that for any $\theta \notin \Theta_I$ and $|m-n| \geqslant |I|/20$,

$$|G_I(z,\theta)(m,n)| \leqslant e^{-\frac{1}{2}L(z)|m-n|} \leqslant e^{-\frac{\log\lambda}{80}|I|}.$$

Applying Theorem 4.6 and Corollary 4.3, for any $\theta \in \mathbb{T}$, we obtain

$$\begin{aligned}\#\big\{|j| \leqslant N :\ \theta + j\alpha \bmod \mathbb{Z} \in \Theta_I\big\} &\lesssim_{\gamma,\eta} (\deg \Theta_I)^C N N^{-1/\gamma} (\log N)^2 \\ &\lesssim_{\gamma,\eta,h} L(z)^{2C} N^{1+5\delta C - \frac{1}{\gamma}} (\log N)^2 \\ &= o(N),\end{aligned}$$

which implies that there exists some $|j'| \leqslant N$ such that $\theta + j'\alpha \bmod \mathbb{Z} \notin \Theta_I$. Consequently,

$$|G_I(z, \theta + j'\alpha)(m,n)| \leqslant e^{-\frac{\log\lambda}{80}|I|}.$$

Moreover, by (20) and $|I_{j'}| = |I|$, it follows that

$$|G_{I_{j'}}(z,\theta)(m,n)| \leqslant e^{-\frac{\log\lambda}{80}|I_{j'}|}.$$

Finally, applying Corollary 3.3 with $\Psi(N) = N^\delta$ where $\delta = \frac{1}{5\gamma C} - \varepsilon$, we conclude

$$\sup_{\theta\in\mathbb{T}} \langle |X_{H_\theta}|_\phi^p \rangle(T) \leqslant (\log T)^{5p\gamma C + \varepsilon}.$$

This completes the proof. □

6.2. **Proof of Theorem 1.2.**

Proof. We present only the essential steps, as the proof is similar to that of Theorem 1.1.

Fix a small $\varepsilon > 0$ and take N sufficiently large. Let $C \geqslant 1$ be the constant $C(d)$ from Lemma 2.5 when $d = 1$. Define an interval I centered at 0 and of length

$$|I| = \exp\left[\delta(\log N)^{1/\gamma}\right]$$

with

$$\delta = \frac{1}{5C}\left[\left(\frac{1}{\kappa}\right)^{1/\gamma} - \varepsilon\right].$$

Applying Theorem 4.6 together with Corollary 4.4, we obtain that for any $\theta \in \mathbb{T}$,

$$\begin{aligned}
&\#\left\{|j| \leqslant N : \theta + j\alpha \bmod \mathbb{Z} \in \Theta_I\right\} \\
&\qquad \lesssim_{\gamma,\kappa,\eta} (\deg \Theta_I)^C N (\log N)^{1+\frac{1}{\gamma}} \exp\left[-\left(\frac{1}{\kappa}\log N\right)^{1/\gamma}\right] \\
&\qquad \lesssim_{\gamma,\kappa,\eta,h} L(z)^{2C} N \exp\left[2\log\log N - \varepsilon(\log N)^{1/\gamma}\right] \\
&\qquad = o(N),
\end{aligned}$$

which implies that there exists $|j'| \leqslant N$ such that

$$|G_{I_{j'}}(z,\theta)(m,n)| \leqslant e^{-\frac{\log\lambda}{80}|I_{j'}|}.$$

Applying Corollary 3.4 with $\Psi(N) = \exp\left[\delta(\log N)^{1/\gamma}\right]$, we conclude

$$\begin{aligned}
\sup_{\theta\in\mathbb{T}} \langle |X_{H_\theta}|_\phi^p\rangle(T) &\leqslant \exp\left[p\left(\frac{5C+\varepsilon}{(1/\kappa)^{1/\gamma}}\log\log T\right)^{\gamma}\right] \\
&\leqslant \exp\Big(p\kappa(5C+\varepsilon)^{\gamma}(\log\log T)^{\gamma}\Big).
\end{aligned}$$

□

6.3. **Proof of Theorem 1.3.**

Proof. We present only the essential steps, as the proof is similar to that of Theorem 1.1.

Denote by $C \geqslant 1$ the constant $C(d)$ from Lemma 2.5 when $d = 1$. Since $\gamma < \frac{1}{5C}$, fix a small $\varepsilon > 0$ such that $5C\gamma + \varepsilon < 1$. Let N be sufficiently large, and define an interval I centered at 0 with length

$$|I| = (\log N)^{1/\delta},$$

where

$$\delta = 5C\gamma + \varepsilon.$$

Applying Theorem 4.6 in combination with Corollary 4.5, we obtain that for any $\theta \in \mathbb{T}$,

$$\begin{aligned}\#\{|j| \leqslant N : \theta + j\alpha \bmod \mathbb{Z} \in \Theta_I\} &\lesssim_{\kappa,\gamma,\eta} (\deg \Theta_I)^C N(\log N)^{-1/\gamma} \\ &\lesssim_{\kappa,\gamma,\eta,h} L(z)^{2C} N(\log N)^{\frac{5C}{\delta}-\frac{1}{\gamma}} \\ &= o(N),\end{aligned}$$

which implies that there exists $|j'| \leqslant N$ such that

$$|G_{I_{j'}}(z,\theta)(m,n)| \leqslant e^{-\frac{\log\lambda}{80}|I_{j'}|}.$$

Applying Corollary 3.5 with $\Psi(N) = (\log N)^{1/\delta}$ yields

$$\sup_{\theta\in\mathbb{T}} \langle |X_{H_\theta}|^p_\phi\rangle(T) \leqslant \exp\left(p(\log T)^{5C\gamma+\varepsilon}\right).$$

□

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References

[Aiz94] M. Aizenman. Localization at weak disorder: Some elementary bounds. *Rev. Math. Phys.*, 6(5A):1163–1182, 1994. Special issue dedicated to Elliott H. Lieb.

[BJ00] J. Bourgain and S. Jitomirskaya. Anderson localization for the band model. In *Geometric aspects of functional analysis*, volume 1745 of *Lecture Notes in Math.*, pages 67–79. Springer, Berlin, 2000.

[BN19] G. Binyamini and D. Novikov. Complex cellular structures. *Ann. of Math. (2)*, 190(1):145–248, 2019.

[Bou05a] J. Bourgain. *Green's function estimates for lattice Schrödinger operators and applications*, volume 158 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2005.

[Bou05b] J. Bourgain. Positivity and continuity of the Lyapounov exponent for shifts on $\mathbb{T}^d$ with arbitrary frequency vector and real analytic potential. *J. Anal. Math.*, 96:313–355, 2005.

[DF22] D. Damanik and J. Fillman. *One-Dimensional Ergodic Schrödinger Operators: I. General Theory*, volume 221. American Mathematical Society, 2022.

[DT97] M. Drmota and R.F. Tichy. *Sequences, discrepancies and applications*, volume 1651 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.

[DT07] D. Damanik and S. Tcheremchantsev. Upper bounds in quantum dynamics. *J. Amer. Math. Soc.*, 20(3):799–827, 2007.

[DT08] D. Damanik and S. Tcheremchantsev. Quantum dynamics via complex analysis methods: general upper bounds without time-averaging and tight lower bounds for the strongly coupled Fibonacci Hamiltonian. *J. Funct. Anal.*, 255(10):2872–2887, 2008.

[Fur97] A. Furman. On the multiplicative ergodic theorem for uniquely ergodic systems. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(6):797–815, 1997.

[Gro87] M. Gromov. Entropy, homology and semialgebraic geometry. *Astérisque*, (145-146):5, 225–240, 1987. Séminaire Bourbaki, Vol. 1985/86.

[GSB99] I. Guarneri and H. Schulz-Baldes. Lower bounds on wave packet propagation by packing dimensions of spectral measures. *Math. Phys. Electron. J.*, 5:Paper 1, 16, 1999.

[GYZ23] L. Ge, J. You, and Q. Zhou. Exponential dynamical localization: criterion and applications. *Ann. Sci. Éc. Norm. Supér. (4)*, 56(1):91–126, 2023.

[HJ19] R. Han and S. Jitomirskaya. Quantum dynamical bounds for ergodic potentials with underlying dynamics of zero topological entropy. *Anal. PDE*, 12(4):867–902, 2019.

[HZ22] R. Han and S. Zhang. Large deviation estimates and Hölder regularity of the Lyapunov exponents for quasi-periodic Schrödinger cocycles. *Int. Math. Res. Not. IMRN*, (3):1666–1713, 2022.

[JKL20] S. Jitomirskaya, H. Krüger, and W. Liu. Exact dynamical decay rate for the almost Mathieu operator. *Math. Res. Lett.*, 27(3):789–808, 2020.

[JL21] S. Jitomirskaya and W. Liu. Upper bounds on transport exponents for long-range operators. *J. Math. Phys.*, 62(7):Paper No. 073506, 9, 2021.

[JL24] S. Jitomirskaya and W. Liu. Universal reflective-hierarchical structure of quasiperiodic eigenfunctions and sharp spectral transition in phase. *J. Eur. Math. Soc. (JEMS)*, 26(8):2797–2836, 2024.

[JLM24] S. Jitomirskaya, W. Liu, and L. Mi. Sharp palindromic criterion for semi-uniform dynamical localization. *arXiv preprint arXiv:2410.21700*, 2024.

[JP22] S. Jitomirskaya and M. Powell. Logarithmic quantum dynamical bounds for arithmetically defined ergodic Schrödinger operators with smooth potentials. In *Analysis at Large: Dedicated to the Life and Work of Jean Bourgain*, pages 173–201. Springer, 2022.

[JS94] S. Jitomirskaya and B. Simon. Operators with singular continuous spectrum. III. Almost periodic Schrödinger operators. *Comm. Math. Phys.*, 165(1):201–205, 1994.

[JZ22] S. Jitomirskaya and S. Zhang. Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schrödinger operators. *J. Eur. Math. Soc. (JEMS)*, 24(5):1723–1767, 2022.

[Las96] Y. Last. Quantum dynamics and decompositions of singular continuous spectra. *J. Funct. Anal.*, 142(2):406–445, 1996.

[Liu22] W. Liu. Quantitative inductive estimates for Green's functions of non-self-adjoint matrices. *Anal. PDE*, 15(8):2061–2108, 2022.

[Liu23] W. Liu. Power law logarithmic bounds of moments for long range operators in arbitrary dimension. *J. Math. Phys.*, 64(3):Paper No. 033508, 11, 2023.

[LP22] M. Landrigan and M. Powell. Fine dimensional properties of spectral measures. *J. Spectr. Theory*, 12(3):1255–1293, 2022.

[LPT+25] W. Liu, M. Powell, Y. Tang, X. Wang, R. Zhang, and J. Zhou. Semi-algebraic discrepancy estimates for multi-frequency shift sequences with applications to quantum dynamics. *preprint*, 2025.

[LPW24] W. Liu, M. Powell, and X. Wang. Quantum dynamical bounds for long-range operators with skew-shift potentials. *arXiv preprint arXiv:2411.00176*, 2024.

[SS23] M. Shamis and S. Sodin. Upper bounds on quantum dynamics in arbitrary dimension. *J. Funct. Anal.*, 285(7):Paper No. 110034, 20, 2023.

[Yom87] Y. Yomdin. C^k-resolution of semialgebraic mappings. Addendum to: "Volume growth and entropy". *Israel J. Math.*, 57(3):301–317, 1987.

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