Connections between common slice obstructions and the Eisermann ribbon obstruction

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Abstract

In this paper, we are interested in a ribbon link obstruction discovered by Eisermann in 2008, which states that the Jones polynomial of an n-component ribbon link is divisible by the Jones polynomial of the n-component unlink. We conjecture that a part of Eisermann's condition follows from the Alexander module having maximal rank, a well-known necessary condition for topological sliceness. We prove that this conjecture holds for special families of links, such as torus links, a large family of satellite links which includes untwisted cables and Bing doubles, and all links whose prime factors each have at most 14 crossings. We also provide a counterexample to a question of Eisermann.

1 Introduction

In 2008, Eisermann discovered an obstruction to ribbonness that arises from the Jones polynomial:

Theorem 1.1 (Eisermann, [5]). *Let $L = K_1 \sqcup \cdots \sqcup K_n$ be an n-component ribbon link and let $V(L)$ be the Jones polynomial of L. Then all of the following hold:*

(a) L has full Jones nullity

(b) L satisfies the following:

$$(V_L/V_{\bigcirc^n})(i) \equiv V_{K_1}(i) \cdots V_{K_n}(i) \mod 32$$

Here, $\bigcirc^n$ is the n-component unlink and $i = \sqrt{-1}$.

We compare this obstruction to several well-known obstructions to sliceness. In particular, we are interested in exploring whether Eisermann's condition follows from conditions related to the linking matrix, signature, and Alexander polynomial:

Fact 1.2. Let L be an n-component topologically slice link. Then all of the following hold:

(a) The linking matrix of L is 0. (See [8, Lemma 8.13]).

(b) The signature of L is 0. (See [7, Corollary 12.3.2]).

(c) L has full Alexander nullity, meaning the Alexander nullity of L equals $n - 1$. (See [7, Corollary 12.3.14]).

Using the SnapPy Python library [4], we were able to verify the following conjecture for all prime links of up to 14 crossings:

Conjecture 1.3. *Let L be a link with $n \geq 2$ components. If L has full Alexander nullity, then*

1. *L has linking matrix 0.*
2. *L has even signature.*
3. *L has full Jones nullity.*

The main goal of this paper is to prove this conjecture for certain special classes of links:

Theorem 1.4. *Conjecture 1.3 holds for the following three classes of links:*

- *Links whose prime factors each have at most 14 crossings (Section 3)*
- *Torus links (Section 4)*
- *Satellites of knots with pattern isotopic to the unlink (Section 5)*

Standard definitions are given in Section 8. Background, including definitions of the Jones polynomial and Alexander module, is given in Section 2. A counterexample to a question of Eisermann is given in Section 6, and ideas for future work are given in Section 7.

2 Background

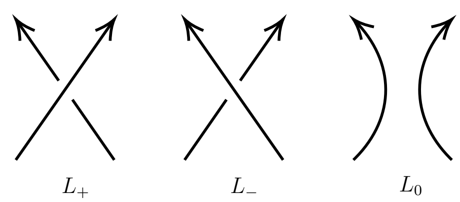
We now go over some background directly related to Conjecture 1.3 so as to fix notation and normalizations that may differ from that of other authors.

2.1 Jones Polynomial

Notation. Following the conventions of [5], given a link L we write the Jones polynomial $V(L) \in \mathbb{Z}[q, q^{-1}]$ in terms of the variable $q = -\sqrt{t}$.

Definition 2.1 (Jones polynomial)**.** The Jones polynomial of a link L is a Laurent polynomial $V_L \in \mathbb{Z}[q, q^{-1}]$ which is invariant under isotopy and satisfies the following Skein relation:

- $V_{\bigcirc}(q) = 1$.
- $(-q + q^{-})V_{L_0}(q) = q^{-2}V_{L_+}(q) - q^2 V_{L_-}(q)$.



Eisermann was interested in the multiplicity of the factor $(q + q^{-1})$ in the Jones polynomial, which we call the **Jones nullity** of a link. The Jones nullity of an n-component link is bounded above by $n - 1$. When this maximum is attained, we say that L has **full Jones nullity**.

In Eisermann's proof of Theorem 1.1, he chose to work with the Kauffman bracket instead of the Jones polynomial:

Definition 2.2 (Kauffman bracket)**.** The Kauffman bracket of a link diagram D is a Laurent polynomial $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ which is invariant under regular isotopy and satisfies the following recursive relations:

- $\langle \bigcirc \rangle = 1$
- $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$.
- $\langle D \sqcup \bigcirc \rangle = \langle D \rangle \cdot (-A^2 - A^{-2})$

The Jones polynomial can be recovered from the Kauffman bracket. If L is a link and D is a diagram for L, then

$$V_L(-A^{-2}) = \langle D \rangle \cdot (-A^{-3})^{\text{writhe}(D)}$$

In particular, we use the fact that the Jones nullity is preserved in the Kauffman bracket and can be read off in the same way. This is because a factor of $(q + q^{-})$ in the Jones polynomial corresponds to a factor of $(-A^{-2} - A^{+2})$ in the Kauffman bracket, and multiplicity of this factor is unaffected by changes in writhe.

2.2 The Alexander module

For every n-component link $L \subseteq S_3$, we can construct a module over the Laurent polynomial ring in n variables $\Lambda = \mathbb{Z}[t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1}]$. This module is called the *Alexander module* of L, and is denoted as $A(L)$. A full construction of the module may be found in [7, Chapter 7]. We provide a brief description here.

Let $E = S^3 \setminus L$. We can consider $\pi_1(E)$ and its abelianization map $\gamma : \pi_1(E) \to H_1(E) \cong \mathbb{Z}^n$. Denote by E_γ a covering space with covering map $p : E_\gamma \to E$ such that $p_*(\pi_1(E_\gamma)) = [\pi_1(E), \pi_1(E)]$, the commutator subgroup. This is known as the *universal abelian cover*. We can then identify $H_1(E)$ with the deck transformation group of p. Fixing a point $b \in E$, we can view $H_1(E_\gamma, p^{-1}(b))$ as a module over the group ring $\mathbb{Z}H_1(E) \cong \Lambda$. This module is the Alexander module of L.

One nice way to think about the universal abelian cover is by considering a Seifert surface bounding each component of the link which acts a "portal" connecting each sheet of our covering space. Then, each variable t_i in Λ acts by "moving through" the portal defined by the ith component of L in the positive orientation (see Figure 1).

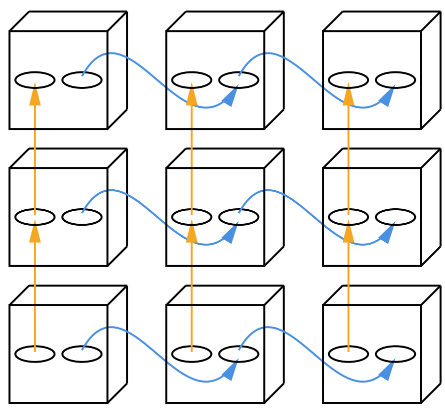


Figure 1: A drawing of the universal abelian cover of the 2-component unlink. Passing through the left component of the link transports you up or down a sheet, while passing through the right component transports you left or right a sheet.

It is often more useful to analyze the Alexander module via a presentation matrix of the module. The most direct way to obtain such a presentation matrix is to perform Fox calculus on the Wirtinger presentation of $\pi_1(S^3 \setminus L)$:

Definition 2.3 (Wirtinger presentation)**.** Let $L \subseteq S^3$ be a link. The link exterior $S^3 \setminus L$ and its fundamental group $\pi_1(S^3 \setminus L)$ are isotopy invariants. Given a link diagram for L, one can construct a group presentation called the Wirtinger presentation for $\pi_1(S^3 \setminus L)$ where each generator corresponds to an arc on the link diagram and each relation corresponds to a crossing. One of these relations will always be redundant, so for a link diagram with p crossings, we can always find a Wirtinger presentation with p generators and $p-1$ relations.

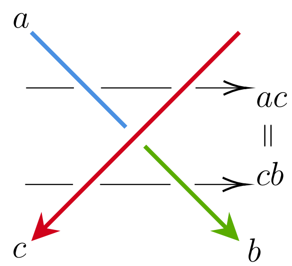


Figure 2: An example of how a relation in the Wirtinger presentation corresponds to a crossing in a link diagram.

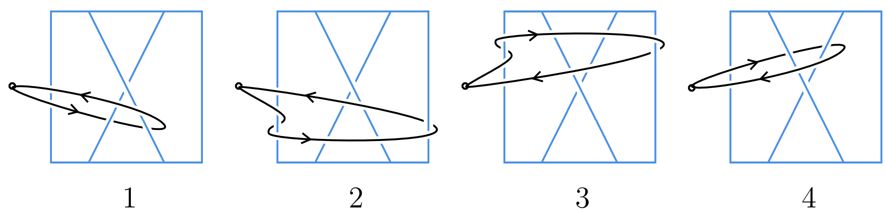


Figure 3: A drawing that shows why the Wirtinger presentation of the fundamental group of a link complement is overdetermined: From left to right, the loop drawn in black can be moved to the other side of the crossing without passing behind the crossing. Thus, the relation corresponding to this crossing is unnecessary. The black loop is allowed to pass through itself.

Definition 2.4 (Fox derivative)**.** Let $G = \langle x_1, \cdots, x_n \rangle$ be a free group. For each $i \in \{1, \cdots, n\}$, the Fox derivative with respect to x_i is the map $\frac{\partial}{\partial x_i} : G \to \mathbb{Z}G$ given by the following rules:

- $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$ (Kronecker delta)
- $\frac{\partial}{\partial x_i}(uv) = \frac{\partial}{\partial x_i}(u) + u\frac{\partial}{\partial x_i}(v)$ for any $u, v \in G$

Theorem 2.5. [7, Theorem 7.1.5] *Let L be a link, and let $\langle x_1, \cdots, x_p \mid r_1, \cdots, r_{p-1} \rangle$ be a presentation for $\pi_1(S^3 \setminus L)$. Let $G = \langle x_1, \cdots, x_p \rangle$, and let $\psi : G \to \pi_1(S^3 \setminus L)$ be the natural quotient map. Let $\gamma : \pi_1(S^3 \setminus L) \to H_1(S^3 \setminus L)$ be the abelianization map. Extend $\gamma \circ \psi : G \to H_1(S^3 \setminus L)$ to a map $\mathbb{Z}G \to \mathbb{Z}H_1(S^3 \setminus L) \cong \Lambda$ by linearity. Then, the $p \times (p-1)$ matrix with (i, j)-th entry $\gamma \circ \psi(\partial r_j / \partial x_i)$ is a presentation matrix for $A(L)$.*

Notation. When we apply Theorem 2.5, we often supress the notation of ψ, simply writing γ when we really mean $\gamma \circ \psi$.

Notation. Many authors, including those of [7] and [6], use the convention that matrices act on the right of row vectors. In this paper we view matrices as acting on the left of column vectors. As such, our matrices are the transpose of how they would be written in those texts. The statement of Theorem 2.5 has been altered to align with our convention.

Here, we restate [7, Definition 7.3.1], which defines the Alexander nullity of a link:

Definition 2.6 (Alexander nullity)**.** The Alexander nullity of an n-component link L, denoted $\beta(L)$, is equal to rank $A(L) - 1$. Equivalently, given a presentation matrix P of $A(L)$ with p rows we have $\beta(L) = p - \text{rank}(P) - 1$, where rank$(P)$ is computed over the field of fractions of Λ. We say that L has **full Alexander nullity** if $\beta(L) = n - 1$. Note that $\beta(L) \leq n - 1$ by [7, Corollary 7.3.13].

3 Links with small prime factors

The goal of this section is to prove Theorem 3.2, which tells us that the property of being a counterexample to Conjecture 1.3 is inherited by at least one of a link's prime factors. Since all prime links of up to 14 crossings have been manually checked via [4] to satisfy Conjecture 1.3, we conclude that Conjecture 1.3 also holds for all composite links arising from these primes (Corollary 3.3).

Lemma 3.1. *Alexander nullity is additive under connected sum. That is, given links* L_1 *and* L_2*, we have* $\beta(L_1 \# L_2) = \beta(L_1) + \beta(L_2)$.

Proof. This proof has three steps:

1. Use the Wirtinger presentation to understand how the fundamental group of the complement of $L_1 \# L_2$ relates to that of L_1 and L_2.

2. Use Fox calculus to obtain a presentation matrix for $A(L_1 \# L_2)$ in terms of presentation matrices of $A(L_1)$ and $A(L_2)$.

3. Compute $\beta(L_1 \# L_2)$ from this presentation matrix.

Step 1: Let n and m denote the number of components in L_1 and L_2, respectively. Fix link diagrams for L_1 and L_2, and let p and q be the number of crossings in each diagram. We consider the Wirtinger presentations for these diagrams:

$$G_1 := \pi_1(S^3 \setminus L_1) = \langle a_1, \ldots, a_p \mid r_1, \ldots, r_{p-1} \rangle,$$
$$G_2 := \pi_1(S^3 \setminus L_2) = \langle b_1, \ldots, b_q \mid s_1, \ldots, s_{q-1} \rangle.$$

Performing a connected sum has the effect of identifying the two generators corresponding to the two arcs merged. Without loss of generality, we may assume these are the generators $a_1 \in G_1$ and $b_1 \in G_2$.

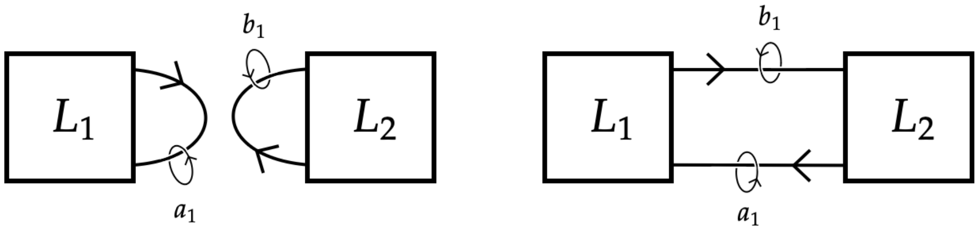


Figure 4: Generators of merged arcs under connected sum

Thus, the Wirtinger presentation of the connected sum is the union of the generators and relations, with the additional relation $a_1 = b_1$:

$$\pi_1(S^3 \setminus (L_1 \# L_2)) = \langle a_1, \dots, a_p, b_1, \dots, b_q \mid r_1, \dots, r_{p-1}, s_1, \dots, s_{q-1}, a_1 b_1^{-1} \rangle.$$

Step 2: Let P_1 and P_2 denote the presentation matrices of the Alexander modules $A(L_1)$ and $A(L_2)$ obtained via Fox calculus. Let the number of rows in the matrices P_1 and P_2 be p and q, respectively. Let e_1 and f_1 denote $p \times 1$ and $q \times 1$ column vectors, respectively, with first entry 1 and all other entries 0. Using Fox calculus on the relations of $\pi_1(S^3 \setminus (L_1 \# L_2))$ yields the following presentation matrix, written in block form:

$$P_{\#} = \begin{bmatrix} P_1 & 0 & e_1 \\ 0 & P_2 & -f_1 \end{bmatrix}$$

Step 3: Our goal is to show that $\mathrm{rank}(P_{\#}) = \mathrm{rank}(P_1) + \mathrm{rank}(P_2) + 1$, since from this it would follow that

$$\begin{aligned} \beta(L_1 \# L_2) &= (p+q) - \mathrm{rank}(P_{\#}) - 1 \\ &= (p+q) - \mathrm{rank}(P_1) - \mathrm{rank}(P_2) - 2 \\ &= (p - \mathrm{rank}(P_1) - 1) + (q - \mathrm{rank}(P_2) - 1) \\ &= \beta(L_1) + \beta(L_2). \end{aligned}$$

This reduces to showing that the column vector $\begin{bmatrix} e_1 \\ -f_1 \end{bmatrix}$ is not a linear combination of the columns of the matrix $\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ (with coefficients from the field of fractions). For this, it suffices to show that e_1 is not a linear combination of the columns of P.

As in Section 2.2, let Λ denote the Laurent polynomial ring in n variables so that the entries of P lie in Λ. Let $\gamma : \mathbb{Z}G_1 \to \Lambda$ denote the abelianization map $G_1 \to H_1(S^3 \setminus L_1) \cong \prod_{i=1}^n \langle t_i \rangle$ extended by linearity to the group rings. By [7, Theorem 7.1.5], we have the following chain complex:

$$\Lambda^{p-1} \xrightarrow{P} \Lambda^p \xrightarrow{\partial_1} \Lambda \longrightarrow 0,$$

where ∂_1 is given by the $1 \times p$ matrix whose ith entry is $\gamma(a_i) - 1$. We may tensor each module in this chain complex with the field of fractions $\mathrm{Frac}(\Lambda)$ to get the chain complex

$$\mathrm{Frac}(\Lambda)^{p-1} \xrightarrow{P} \mathrm{Frac}(\Lambda)^p \xrightarrow{\partial_1} \mathrm{Frac}(\Lambda) \longrightarrow 0.$$

Note that $\partial_1(e_1) = \gamma(a_1) - 1$. Since a_1 is a generator of G_1, we have that $\gamma(a_1)$ is a nontrivial element of $H_1(S^3 \setminus L_1) \subseteq \Lambda$. In particular, $\gamma(a_1) \neq 1$. So $e_1 \notin \ker \partial_1$, meaning $e_1 \notin \mathrm{im}(P)$. Thus, e_1 is not in the column space of P, and we are done. □

Theorem 3.2. *Suppose that L is a link for which Conjecture 1.3 does not hold. Then Conjecture 1.3 must fail to hold for at least one prime factor of L.*

Proof. This follows from the behavior of our link invariants under connected sum. Suppose $L = L_1 \# L_2$. Then

1. L has linking matrix 0 if and only if both L_1 and L_2 have linking matrix 0.

2. Signature is additive under connected sum. In particular, if L has odd signature, then either L_1 or L_2 has odd signature.

3. Jones polynomial is multiplicative under connected sum. In particular, Jones nullity is additive under connected sum, meaning L has full Jones nullity if and only if both L_1 and L_2 have full Jones nullity.

4. Alexander nullity is additive under connected sum by Lemma 3.1. In particular, L has full Alexander nullity if and only if both L_1 and L_2 have full Alexander nullity.

□

Corollary 3.3. *Every link whose prime decomposition consists solely of links with at most* 14 *crossings satisfies Conjecture 1.3.*

4 Results on torus links

We have seen that all prime links of up to 14 crossings satisfy Conjecture 1.3. Here, we consider the special case of torus links:

Definition 4.1 (Torus knot)**.** Let $K_1 \sqcup K_2$ be the Hopf link, and let $N(K_1)$ be a tubular neighborhood of K_1 disjoint from K_2. For coprime integers p and q, we define the (p, q) torus knot, denoted $T(p, q)$, to be the knot lying on the boundary of $N(K_1)$ (a torus) for which $\mathrm{lk}(T(p, q), K_1) = q$ and $\mathrm{lk}(T(p, q), K_2) = p$.

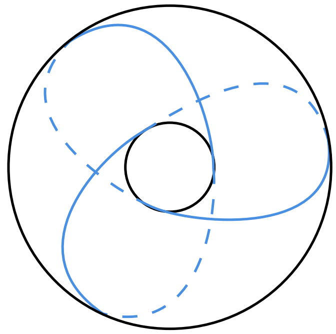


Figure 5: The trefoil, $T(2,3)$, lying on a torus.

Definition 4.2 (Torus link)**.** [1] Let p and q be coprime integers, and let n be a positive integer. Then we define the (p,q) torus n-link, denoted $T(np,nq)$, to be the link consisting of n parallel copies of $T(p,q)$ lying on a torus embedded in S^3.

When $p,q \neq 0$ and $n \geq 2$, the link $T(np,nq)$ has nonzero linking matrix. Our conjecture would then suggest that these links do not have full Alexander nullity. We find that this is true:

Theorem 4.3. *Let p and q be nonzero coprime integers, and let $n \geq 2$. Then the torus link $T(np,nq)$ has Alexander nullity 0.*

Proof. By [1, Theorem 4.3] we have that

$$\pi_1(S^3 \setminus T(np,nq)) = \langle \alpha, \beta, f_1, \cdots, f_{n-1} \mid \alpha^p\beta^{-q}, \alpha^p f_1 \beta^{-q} f_1^{-1}, \cdots, \alpha^p f_{n-1}\beta^{-q} f_{n-1}^{-1} \rangle.$$

If the generators are made to commute with each other, all relations reduce to the first one. So $H_1(S^3 \setminus T(np,nq))$ is the free abelian group on the generators $\alpha, \beta, f_1, \cdots, f_{n-1}$ with the additional relation $\alpha^p = \beta^q$. Since $T(np,nq)$ is an n-component link, this is isomorphic to the free abelian group $\langle t_1 \rangle \times \cdots \times \langle t_n \rangle$. An explicit isomorphism γ is given by

$$\begin{aligned} \gamma(\alpha) &= t_n^q, \\ \gamma(\beta) &= t_n^p, \\ \gamma(f_i) &= t_i \qquad \text{for all } i \in \{1, \cdots, n-1\}. \end{aligned}$$

Using the method of Fox free calculus (Theorem 2.5) we find that the following is an $(n+1) \times n$ presentation matrix for the Alexander module of $T(np,nq)$:

$$\begin{bmatrix} \frac{t_n^{pq}-1}{t_n^q-1} & \frac{t_n^{pq}-1}{t_n^q-1} & \frac{t_n^{pq}-1}{t_n^q-1} & \cdots & \frac{t_n^{pq}-1}{t_n^q-1} \\ -\frac{t_n^{pq}-1}{t_n^p-1} & -t_1\frac{t_n^{pq}-1}{t_n^p-1} & -t_2\frac{t_n^{pq}-1}{t_n^p-1} & \cdots & -t_{n-1}\frac{t_n^{pq}-1}{t_n^p-1} \\ 0 & t_n^{pq}-1 & 0 & \cdots & 0 \\ 0 & 0 & t_n^{pq}-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_n^{pq}-1 \end{bmatrix}.$$

This matrix has $n+1$ rows and rank n, meaning the Alexander nullity is $(n+1)-n-1=0$. □

5 Results on satellite links

Here we characterize a large class of satellite links that have full Alexander nullity and even signature.

Definition 5.1 (Satellite link)**.** Let $K \subseteq S^3$ be a knot, and let $L \subseteq S^1 \times D^2$ be a link embedded in a solid torus. Let c denote a point in the interior of D^2 and let p denote a point on the boundary of D^2. Let $\varphi : S^1 \times D^2 \to S^3$ be an embedding of the solid torus into S^3 such that $\varphi(S^1 \times \{c\}) = K$ and $\mathrm{lk}(\varphi(S^1 \times \{c\}), \varphi(S^1 \times \{p\})) = 0$. Then we call the link $\varphi(L)$ a satellite of K with pattern L.

Proposition 5.2. *Fix a diagram for a link L. Then $\pi_1(S^3 \setminus L)$ is generated by the set of meridians around arcs containing local maxima in the diagram.*

Proof. This is well-known and follows directly from considering a Wirtinger presentation for the diagram. See, for example, [2, Introduction]. □

Theorem 5.3. *Let K be a knot, let $P = K_0 \sqcup K_1 \sqcup \cdots \sqcup K_n$ be an $n+1$ component link with the following properties:*

1. *K_0 is isotopic to the unknot.*
2. *$P \setminus K_0$ is isotopic to the n-component unlink.*

$S^3 \setminus K_0$ is homeomorphic to a solid torus, so viewing $P \setminus K_0$ as a subset of this solid torus, let L denote the satellite of K with pattern $P \setminus K_0$. Then L has full Alexander nullity.

Proof. See Figure 6. This proof has two steps:

1. Compute a group presentation for $\pi_1(S^3 \setminus L)$ using Van Kampen's theorem.
2. Use Fox calculus to compute a presentation matrix for the Alexander module of L, then compute its rank.

Step 1: Let A be an open tubular neighborhood of K containing L, and let B be the interior of the complement of A. Make A slightly thicker so that $A \cup B = S^3 \setminus L$. In order to apply Van Kampen's theorem, we make the following observations:

- By applying Proposition 5.2 to a suitable diagram of P, we find that there exists a presentation $\langle t_1, \cdots, t_d, a_1, \cdots, a_n \mid s_1, \cdots, s_{n+d-1}\rangle$ for $\pi_1(S^3 \setminus P)$ such that a_i is a meridian of K_i for all $i \in \{1, \cdots, n\}$ and t_i is a meridian of K_0 for all $i \in \{1, \cdots, d\}$. The subspace A is homeomorphic to $S^3 \setminus P$, so we may write

$$\pi_1(A) = \pi_1(S^3 \setminus P) = \langle t_1, \cdots, t_d, a_1, \cdots, a_n \mid s_1, \cdots, s_{n+d-1}\rangle.$$

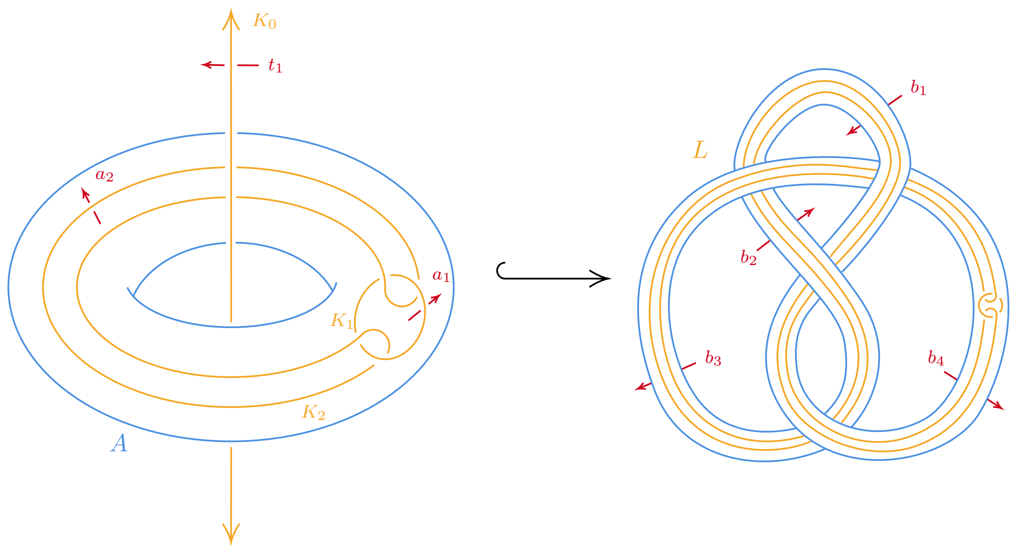


Figure 6: An illustration of how things will be labeled.

- Fix a Wirtinger presentation $\langle b_1, \cdots, b_p \mid r_1, \cdots, r_{p-1}\rangle$ for K. The subspace B is a deformation retract of $S^3 \setminus K$, so we may write

$$\pi_1(B) = \pi_1(S^3 \setminus K) = \langle b_1, \cdots, b_p \mid r_1, \cdots, r_{p-1}\rangle.$$

- $A \cap B$ deformation retracts to a torus, so $\pi_1(A \cap B) = \langle m \rangle \times \langle \ell \rangle$, where m represents a meridian on the torus and ℓ represents a longitude. Let ι^A denote the inclusion $A \cap B \hookrightarrow A$, and let ι^A_* denote the induced map on the fundamental groups. Define ι^B_* analogously.

By Van Kampen's theorem, we have

$$\pi_1(S^3 \setminus L) = \langle b_1, \cdots, b_p, t_1, \cdots, t_d, a_1, \cdots, a_n \mid r_1, \cdots, r_{p-1}, s_1, \cdots, s_{n+d-1}, \\ \iota^A_*(m)\iota^B_*(m)^{-1}, \iota^A_*(\ell)\iota^B_*(\ell)^{-1}\rangle.$$

Step 2: Let $\gamma : \pi_1(S^3 \setminus L) \to H_1(S^3 \setminus L)$ be the abelianization map. Since each a_i is a meridian of K_i and L is the image of an embedding of $K_1 \sqcup \cdots \sqcup K_n$ into S^3, we have that $H_1(S^3 \setminus L) = \langle \gamma(a_1)\rangle \times \cdots \times \langle \gamma(a_n)\rangle$.

Before we use Fox calculus to write down a presentation matrix for the Alexander module of L, we first argue that some blocks of this matrix consist of all zeros:

- Claim: For all $i \in \{1, \cdots, n\}$ and $j \in \{1, \cdots, n+d-1\}$, we have $\gamma(\frac{\partial s_j}{\partial a_i}) = 0$.

  Proof: Notice that $\gamma(t_j) = 1$ for all $j \in \{1, \cdots, d\}$ since the loop t_j has linking number 0 with K_i for all $i \in \{1, \cdots, n\}$. Therefore,

$$\gamma\left(\frac{\partial s_j}{\partial a_i}\right) = \gamma\left(\frac{\partial s_j}{\partial a_i}\Big|_{t_1, \cdots, t_d = 1}\right) = \gamma\left(\frac{\partial (s_j|_{t_1,\cdots,t_d=1})}{\partial a_i}\right).$$

It now suffices to show that $s_j|_{t_1,\cdots,t_d=1} = 1$. Recall that $P \setminus K_0$ is the n-component unlink, meaning

$$\pi_1(S^3 \setminus (P \setminus K_0)) = \langle a_1, \cdots, a_n \rangle.$$

Let $\varphi : S^3 \setminus P \hookrightarrow S^3 \setminus (P \setminus K_0)$ be the natural inclusion map, and let

$$\varphi_* : \pi_1(S^3 \setminus P) = \langle t_1, \cdots, t_d, a_1, \cdots, a_n \mid s_1, \cdots, s_{n+d-1} \rangle \\ \to \langle a_1, \cdots, a_n \rangle = \pi_1(S^3 \setminus (P \setminus K_0))$$

be the induced map on the fundamental groups. Intuitively, φ_* describes what happens to the fundamental group of $S^3 \setminus P$ when we "fill in" K_0. From this description it is clear that φ_* maps meridians of K_0 to the identity. In particular, $\varphi_*(t_i) = 1$ for all $i \in \{1, \cdots, d\}$.

Let $\psi : \langle t_1, \cdots, t_d, a_1, \cdots, a_n \rangle \to \pi_1(S^3 \setminus P)$ be the natural projection from the free group on $n + d$ generators. Then the composition $\varphi_* \circ \psi$ is exactly the evaluation homomorphism $|_{t_1,\cdots,t_d=1}$. In other words, the following diagram commutes:

$$\begin{array}{ccc} \langle t_1, \cdots, t_d, a_1, \cdots, a_n \rangle & \xrightarrow{\psi} & \langle t_1, \cdots, t_d, a_1, \cdots, a_n \mid s_1, \cdots, s_{n+d-1} \rangle \\ & \searrow^{|_{t_1,\cdots,t_d=1}} & \downarrow \varphi_* \\ & & \langle a_1, \cdots, a_n \rangle \end{array}$$

Since $\psi(s_j) = 1$, we have that $s_j|_{t_1,\cdots,t_d=1} = \varphi_* \circ \psi(s_j) = \varphi(1) = 1$, as desired.

- Claim: For all $i \in \{1, \cdots, n\}$, we have $\gamma(\frac{\partial(\iota_*^A(\ell)\iota_*^B(\ell)^{-1})}{\partial a_i}) = 0$.

  Proof: First notice that $\frac{\partial(\iota_*^A(\ell)\iota_*^B(\ell)^{-1})}{\partial a_i} = \frac{\partial \iota_*^A(\ell)}{\partial a_i}$ since the word $\iota_*^B(\ell)^{-1}$ does not contain a_i. Let φ and ψ be defined as before. By the same reasoning as in the previous claim, it suffices to show that $\varphi_* \circ \psi(\iota_*^A(\ell)) = 1$.

  Recall that ℓ is a longitude of the torus $A \cap B$. Pulling ℓ back along the homeomorphism $S^3 \setminus P \cong A$, we find that ℓ corresponds to a meridian of K_0. It follows that $\psi(\iota_*^A(\ell))$ is a meridian of K_0. Since φ_* sends meridians of K_0 to the identity, we have $\varphi_* \circ \psi(\iota_*^A(\ell)) = 1$, as desired.

With these facts in mind, by Theorem 2.5 we get the following presentation matrix for the Alexander module of L, writing $[m]$ as shorthand for $\iota_*^A(m)\iota_*^B(m)^{-1}$ and $[\ell]$ as shorthand for $\iota_*^A(\ell)\iota_*^B(\ell)^{-1}$:

$$\begin{array}{c} \\ b_1 \\ \vdots \\ b_p \\ t_1 \\ \vdots \\ t_d \\ a_1 \\ \vdots \\ a_n \end{array}
\begin{array}{c} \begin{array}{cccccccc} r_1 & \cdots & r_{p-1} & s_1 & \cdots & s_{n+d-1} & [m] & [\ell] \end{array} \\
\begin{bmatrix}
\gamma(\frac{\partial r_1}{\partial b_1}) & \cdots & \gamma(\frac{\partial r_{p-1}}{\partial b_1}) & 0 & \cdots & 0 & \gamma(\frac{\partial([m])}{\partial b_1}) & \gamma(\frac{\partial[\ell]}{\partial b_1}) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\gamma(\frac{\partial r_1}{\partial b_p}) & \cdots & \gamma(\frac{\partial r_{p-1}}{\partial b_p}) & 0 & \cdots & 0 & \gamma(\frac{\partial[m]}{\partial b_p}) & \gamma(\frac{\partial[\ell]}{\partial b_p}) \\
0 & \cdots & 0 & \gamma(\frac{\partial s_1}{\partial t_1}) & \cdots & \gamma(\frac{\partial s_{n+d-1}}{\partial t_1}) & \gamma(\frac{\partial[m]}{\partial t_1}) & \gamma(\frac{\partial[\ell]}{\partial t_1}) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \gamma(\frac{\partial s_1}{\partial t_d}) & \cdots & \gamma(\frac{\partial s_{n+d-1}}{\partial t_d}) & \gamma(\frac{\partial[m]}{\partial t_d}) & \gamma(\frac{\partial[\ell]}{\partial t_1}) \\
0 & \cdots & 0 & 0 & \cdots & 0 & \gamma(\frac{\partial[m]}{\partial a_1}) & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \gamma(\frac{\partial[m]}{\partial a_n}) & 0
\end{bmatrix} \end{array}.$$

The number of rows in this matrix is $p+d+n$. We claim that this matrix has rank at most $p+d$. If this were the case, then $\beta(L) \geq (p+d+n)-(p+d)-1 = n-1$, and we would be done.

For this proof, we will interpret rank as "the dimension of the row space". If $\gamma(\frac{\partial[m]}{\partial a_i}) = 0$ for all i then the matrix clearly has the desired rank, so from now on, assume without loss of generality that $\gamma(\frac{\partial[m]}{\partial a_1}) \neq 0$.

The rows labeled $a_2, \cdots, a_n$ are multiples of the row labeled a_1, so the set consisting of the rows labeled $b_1, \cdots, b_p, t_1, \cdots, t_d, a_1$ is a spanning set for the row space. Therefore, the rank is at most $p+d+1$. To lower this bound to $p+d$, we will show that this set is linearly dependent. By adding multiples of row a_1 to rows $b_1, \cdots, b_p$, we can eliminate the entries in column $[m]$. To show linear dependence, it now suffices to show that

$$\det \begin{bmatrix} \gamma(\frac{\partial r_1}{\partial b_1}) & \cdots & \gamma(\frac{\partial r_{p-1}}{\partial b_1}) & \gamma(\frac{\partial[\ell]}{\partial b_1}) \\ \vdots & \ddots & \vdots & \vdots \\ \gamma(\frac{\partial r_1}{\partial b_p}) & \cdots & \gamma(\frac{\partial r_{p-1}}{\partial b_p}) & \gamma(\frac{\partial[\ell]}{\partial b_p}) \end{bmatrix} = 0. \tag{1}$$

Notice that γ has the following three properties:

1. γ maps all b_i to the same element in $H_1(S^3 \setminus L)$. (this element can be written as $\prod_{i=1}^n \gamma(a_i)^{\mathrm{lk}(K_0,K_i)}$, but we don't need this fact for the proof).

2. $\gamma(r_i) = 1$ for all $i \in \{1, \cdots, p-1\}$ since relations in a Wirtinger presentation are all of the form $b_j b_k b_j^{-1} b_l^{-1}$.

3. $\gamma([\ell]) = 1$. To see this, we write

   $$\gamma([\ell]) = \gamma(\iota_*^A(\ell)\iota_*^B(\ell)^{-1}) = \gamma(\iota_*^A(\ell))\gamma(\iota_*^B(\ell))^{-1}.$$

   $\gamma(\iota_*^A(\ell)) = 1$ since ℓ has linking number 0 with all K_i, and $\gamma(\iota_*^B(\ell)) = 1$ since ℓ has linking number 0 with K.

By the discussion under [6, Equation 3.6], these three facts imply that

$$\sum_{i=1}^{p} \gamma(\tfrac{\partial[\ell]}{\partial b_i}) = 0 \quad \text{and} \quad \sum_{i=1}^{p} \gamma(\tfrac{\partial r_j}{\partial b_i}) = 0 \quad \text{for all } j \in \{1, \cdots, p-1\}.$$

This tells us that the sum of the rows of the matrix in (1) is zero, meaning the rows are linearly dependent and the determinant is zero, as desired. □

Theorem 5.4. *Let K be a knot, let P be a link embedded in a solid torus in S^3, and let L be a satellite of K with pattern P. Then the signature of L has the same parity as the signature of P. In particular, since the n-component unlink has signature 0, all links satisfying the hypothesis of Theorem 5.3 have even signature.*

Proof. By the construction in [8, Theorem 6.15], L has a Seifert matrix of the form

$$\begin{bmatrix} M & 0 \\ 0 & X \end{bmatrix}$$

where M is a Seifert matrix for P and

$$X = \begin{bmatrix} A & A & A & \cdots & A \\ A^T & A & A & \cdots & A \\ A^T & A^T & A & \cdots & A \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^T & A^T & A^T & \cdots & A \end{bmatrix},$$

where A is a Seifert matrix for K. Writing σ to denote the signature of a link and sig to denote the signature of a matrix, we have

$$\sigma(L) = \text{sig}(M + M^T) + \text{sig}(X + X^T) = \sigma(P) + \text{sig}(X + X^T).$$

Thus, to prove the claim, it suffices to show that $\text{sig}(X + X^T)$ is even. Let

$$B = \begin{bmatrix} I & -I & 0 & \cdots & 0 \\ 0 & I & -I & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{bmatrix},$$

where I denotes the identity matrix of the same size as A, and let

$$Y := B(X + X^T)B^T = \begin{bmatrix} & A - A^T & & & \\ -A + A^T & & A - A^T & & \\ & -A + A^T & & \ddots & \\ & & \ddots & & A - A^T \\ & & & -A + A^T & A + A^T \end{bmatrix}.$$

By Sylvester's law of inertia, $\operatorname{sig}(X + X^T)$ is equal to $\operatorname{sig}(Y)$. Suppose that Y is w blocks wide. To show that Y has even signature, we proceed by induction on w. There are two base cases:

- If $w = 1$ then $Y = \begin{bmatrix} A + A^T \end{bmatrix}$, and $\operatorname{sig}(A + A^T) = \sigma(K)$ which is even since K is a knot.
- If $w = 2$ then $Y = \begin{bmatrix} 0 & A - A^T \\ -A + A^T & A + A^T \end{bmatrix}$. The determinant of this matrix equals $\det(A - A^T)^2$, which is nonzero since $\det(A - A^T) = \Delta_K(1) = \pm 1$ (see [8, Theorem 6.10]). Since Y is even-dimensional and has no zero eigenvalues, we conclude that it must have even signature.

Now assume $w > 2$. Let

$$C = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ I & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}$$

so that

$$CYC^T = \begin{bmatrix} & A - A^T & & & \\ -A + A^T & & 0 & & \\ & 0 & & \ddots & \\ & & \ddots & & A - A^T \\ & & & -A + A^T & A + A^T \end{bmatrix}.$$

This is a block diagonal matrix, so its signature is the sum of the signatures of the blocks. The upper left block has even signature by the same argument as in our second base case, and the lower right block has even signature by our inductive hypothesis. We conclude that $\operatorname{sig}(CYC^T) = \operatorname{sig}(Y) = \operatorname{sig}(X + X^T)$ is even, and we are done. □

Lemma 5.5. *Let L be a 2-component link with even signature. Then $q + q^{-1}$ divides $V_L(q)$. In other words, L has full Jones nullity.*

Proof. Fix a Seifert surface for L and let g be its genus. Since L has two components, the Seifert matrix A has dimension $2g + 1$, which is odd. Since $A + A^T$ has even signature, it must have an odd number of 0 eigenvalues, counting multiplicity. In particular we have $\det(L) = 0$, which implies $V_L(i) = 0$ since $\det(L) = V_L(i)$. Therefore, $V_L(q)$ is divisible by $q + q^{-1}$, the generator of the ideal consisting of polynomials in $\mathbb{Z}[q, q^{-1}]$ which evaluate to 0 at i. □

Corollary 5.6. *Let L be a 2-component link satisfying the hypothesis of Theorem 5.3. Then L has full Jones nullity.*

Proof. Theorem 5.4 tells us that L has even signature. Then Lemma 5.5 tells us that L has full Jones nullity, as desired. ☐

We present applications of these theorems to several well-known classes of links:

5.1 Cable links

Definition 5.7 (Cable link)**.** Let K be a knot, and let $T(p,q) \subseteq S^1 \times D^2$ be a torus link lying on the surface of a solid torus. We define a (p,q) cable of K to be a satellite of K with pattern $T(p,q)$. See Figure 7.

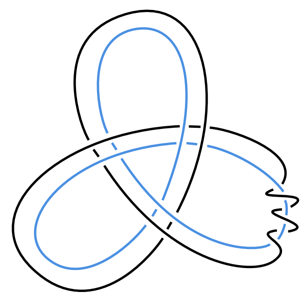
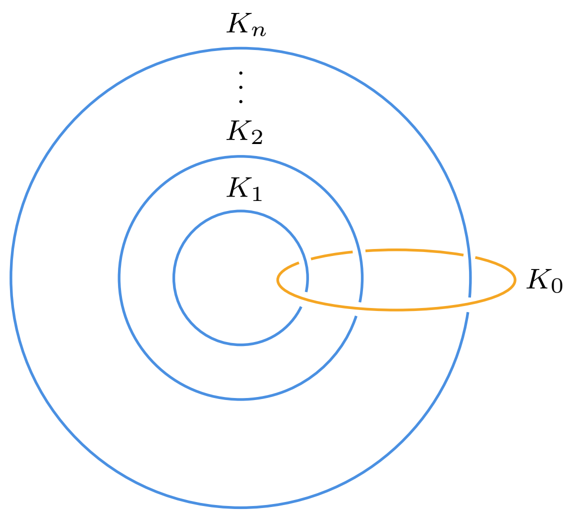


Figure 7: A $(2,0)$ cable of the trefoil.

Corollary 5.8. *Let K be a knot, and let L be an $(n,0)$ cable of K. Then L has full Alexander nullity.*

Proof. In the language of Theorem 5.3, L arises from setting P equal to the following link:



☐

Theorem 5.4 tells us that all $(n, 0)$ cable links have even signature. We can strengthen this result:

Theorem 5.9. *Let K be a knot, and let L be an $(n, 0)$ cable of K. If n is even, then L has signature 0. If n is odd, then L has signature equal to that of K.*

Proof. Since L has linking matrix zero, [9, Theorem 1] tells us that the signature of L is invariant under switching the orientation of any component of L. Thus, for the purpose of computing the signature, we may view L as an unoriented link.

If n is even, then consider the Seifert surface for L consisting of $n/2$ disjoint parallel annuli. (we don't require the surface to be connected, see [3, Section 2]).

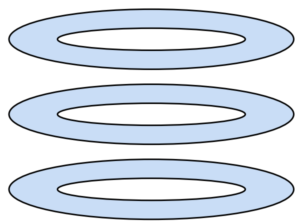


Figure 8: Seifert surface of the $(6, 0)$ cable of the unknot.

The first homology of such a surface has $n/2$ generators, and they all have linking number 0 with each other and with their positive pushes (see Definition 8.4). Thus, the Seifert matrix corresponding to this surface is the zero matrix, which tells us the signature is zero.

If n is odd, then fix a Seifert surface Σ for K and let $K_1, \cdots, K_{n-1}$ be positive pushes of $K = \partial\Sigma$. Since K_1 does not pass through Σ, we have that $\mathrm{lk}(K_1, K) = 0$. By applying the same reasoning to all other pairs, we find that the n-component link $K \sqcup K_1 \sqcup \cdots \sqcup K_{n-1}$ has linking matrix zero. Therefore, this link is an $(n, 0)$ cable of K, and we may take this to be L. Now, we can construct a Seifert surface for L consisting of $\frac{n-1}{2}$ parallel bands with boundary $K_1 \sqcup \cdots \sqcup K_{n-1}$, together with Σ which has boundary K. Call this surface $\overline{\Sigma}$.

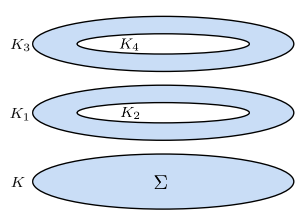


Figure 9: Seifert surface of the $(5, 0)$ cable of the unknot.

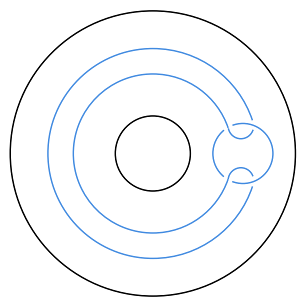
If $H_1(\Sigma)$ has d generators, then $H_1(\overline{\Sigma})$ has $\frac{n-1}{2}$ additional generators. Each of these additional generators has linking number zero with all other generators and with their own positive pushes. Therefore, letting A_Σ and $A_{\overline{\Sigma}}$ denote Seifert matrices for Σ and $\overline{\Sigma}$, respectively, we find that $A_{\overline{\Sigma}}$ is a square matrix of size $d + \frac{n-1}{2}$ which contains a copy of A_Σ in the upper left, and zeros everywhere else. It follows that $A_\Sigma + A_\Sigma^T$ and $A_{\overline{\Sigma}} + A_{\overline{\Sigma}}^T$ have equal signatures, meaning the links K and L have equal signatures. □

Theorem 5.10. *Let K be a knot, and let L be an $(n, 0)$ cable of K. Then $(q+q^{-1})^{n-1}$ divides $V_L(q)$. In other words, L has full Jones nullity.*

Proof. WIP □

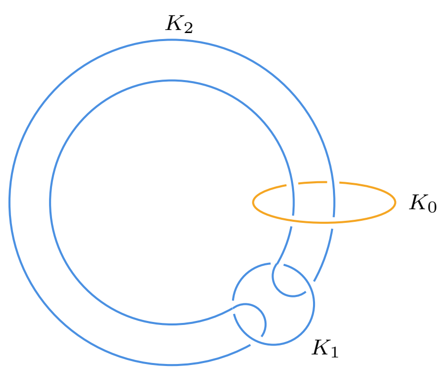
5.2 Bing doubles

Definition 5.11 (Bing double)**.** We define the Bing double of a knot K to be a satellite of K with the following pattern:



Corollary 5.12. *Let K be a knot, and let L be a Bing double of K. Then L has full Alexander nullity.*

Proof. In the language of Theorem 5.3, L arises from setting P equal to the Borromean rings:

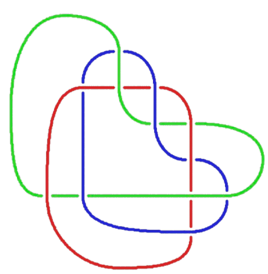


□

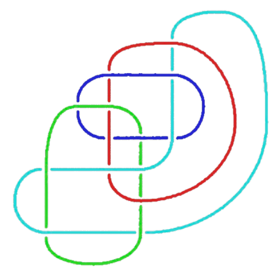
6 Comparing Murasugi nullity to Jones nullity

Since we're exploring the relationship between the Alexander nullity and the Jones nullity, one may also ask the question of whether the Murasugi nullity has a similar relation to the Jones nullity. Indeed, in [5, Question 7.9], Eisermann asked the question of whether the Jones nullity was equal to the Murasugi nullity. Using [4], we have found the following minimal counterexamples, and showed that there is no inequality in either direction:

- The 3-component prime link "L12n1998" has Jones nullity 1 and Murasugi nullity 2. (it also has Alexander nullity 0).
- The 4-component prime link "L14n63006" has Jones nullity 2 and Murasugi nullity 1. (it also has Alexander nullity 0).



L12n1998



L14n63006

The names of these links come from the Thistlethwaite link table.

7 Continuing work

We present some ideas on how to attack the general case of Conjecture 1.3:

Question 7.1. Is Alexander nullity a lower bound for Jones nullity?

This is true for all prime links of up to 14 crossings by [4]. If this were true for all links, then full Alexander nullity would imply full Jones nullity, which is the third part of Conjecture 1.3. If true, this would imply that all topologically slice links have full Jones nullity.

An alternative approach to showing that all slice links have full Jones nullity would be to answer the following:

Question 7.2. Is Jones nullity a concordance invariant?

An idea to prove this would be to follow a similar argument as in the proof of [5, Theorem 1], using induction on the Kauffman bracket skein relations on the concordance surface to reach the desired result.

8 Standard definitions

Definition 8.1 (Regular isotopy)**.** We say two link diagrams are regular isotopic if they differ by a sequence of only the second and third Reidemeister moves.

Definition 8.2 (Linking matrix)**.** Given a link $L = K_1 \sqcup \cdots \sqcup K_n$, the linking matrix of L is the $n \times n$ matrix whose entry in the ith row and jth column equals $\mathrm{lk}(K_i, K_j)$, the linking number between the ith and jth components.

Definition 8.3 (Seifert surface)**.** A Seifert surface of a link L is any orientable surface embedded in S^3 whose boundary is L.

Definition 8.4 (Seifert matrix)**.** Let Σ denote a Seifert surface for a link L. Consider generators of the first homology group, $H_1(S)$, as loops in Σ. Now consider the 3-dimensional manifold $\Sigma \times [0,1]$ embedded in S^3. We can identify a loop α with $\alpha \times \{0\}$ and denote its *positive push* $\alpha \times \{1\}$ by α^+. The Seifert matrix of S is then the matrix whose (i,j) entry is $\mathrm{lk}(\alpha_i, \alpha_j^+)$, the linking number between the ith generator and the positive push of the jth generator.

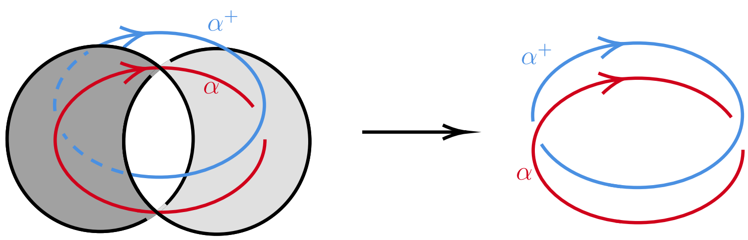


Figure 10: A positive push of a loop on a Seifert surface for the Hopf link

Definition 8.5 (Signature)**.** Recall that the signature of a symmetric matrix is defined to equal the number of positive minus the number of negative eigenvalues, counting multiplicities. We define the signature of a link L, denoted $\sigma(L)$, to equal the signature of the matrix $A + A^T$, where A is a Seifert matrix for L.

Definition 8.6 (Determinant)**.** Let A be a Seifert matrix for a link L. Following the conventions of [5], we define the signed determinant of L, denoted $\det(L)$, to equal the determinant of the matrix $-i(A + A^T)$.

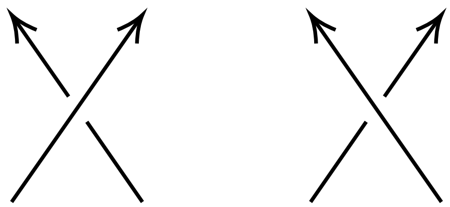
Definition 8.7 (Murasugi nullity)**.** Let A be a Seifert matrix for a link L. We define the Murasugi nullity of L to be the nullity of the matrix A.

Definition 8.8 (Connected sum of links)**.** Let L_1 and L_2 be oriented links. We define $L_1 \# L_2$ to be the result of performing the connected sum operation on some component of L_1 and some component of L_2. In general, different choices of components may result in non-isotopic links. For our purposes, $L_1 \# L_2$ may be taken to be any one of these choices.

Definition 8.9 (Split link)**.** A link L is called split if it is isotopic to a disjoint union of two links that share no crossings.

Definition 8.10 (Prime link)**.** A non-split link L is prime if it is not isotopic to a connected sum of two nontrivial links.

Definition 8.11 (Writhe)**.** The writhe of an oriented link L is the number of positive crossings minus the number of negative crossings. A crossing is determined to be positive or negative based on the following rule:



The left figure is a positive crossing and the right figure is a negative crossing.

References

[1] P. C. Argyres and D. P. Kulkarni. Fundamental groups for torus link complements, arXiv:1904.10005 (2019)

[2] R. Blair, A. Kjuchukova, R. Velazquez, P. Villanueva. Wirtinger Systems of Generators of Knot Groups, *Communications in Analysis and Geometry* 28(2): 243–262 (2020)

[3] A. Conway. The Levine-Tristram Signature: A Survey, arXiv:1904.10005 (2019)

[4] M. Culler, N. M. Dunfield, M. Goerner, and J. R. Weeks, SnapPy, a computer program for studying the geometry and topology of 3-manifolds, http://snappy.computop.org

[5] M. Eisermann. The Jones polynomial of ribbon links, *Geometry & Topology* 13 (2009)

[6] R. H. Crowell and R. H. Fox. *Introduction to Knot Theory*, Springer-Verlag (1963)

[7] A. Kawauchi. *A Survey of Knot Theory*, Birkhäuser Basel (1996)

[8] W. B. R. Lickorish. *An Introduction to Knot Theory*, Springer (1997)

[9] K. Murasugi. On the Signature of Links, *Topology* 9 (1969)