Knot Theory: Connections between common slice obstructions and the Eisermann ribbon obstruction

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Background

Definition

A **knot** is a smooth embedding of S^1 into S^3. An n-component **link** is a smooth embedding of a disjoint union of n circles into S^3.

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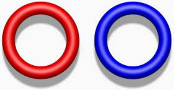

2-component unlink

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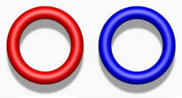
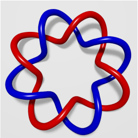


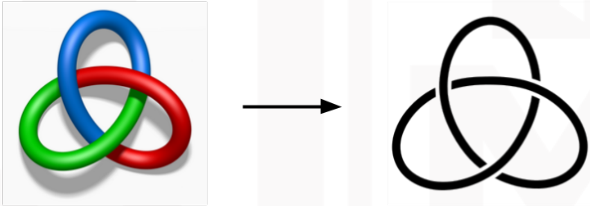
2-component unlink

Trefoil knot



2-component torus link

We typically use **planar diagrams** to depict knots.



Planar diagram of the trefoil knot

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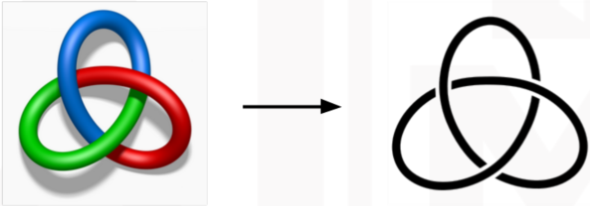


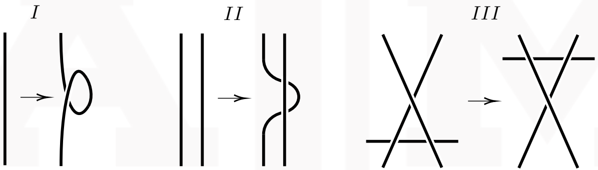
Planar diagram of the trefoil knot

Link Equivalence

What does it mean for two links to be the same?

Definition

Two links are **isotopic** if there is a way to deform one link into the other without passing the strands through each other.

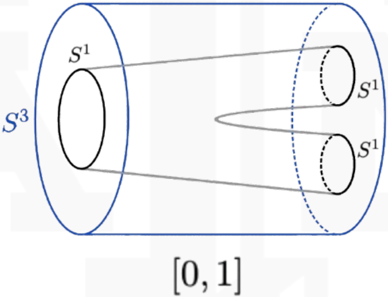


Reidemeister moves

Cobordism

Definition

A **cobordism** between two links L_1 and L_2 is a surface in $S^3 \times [0,1]$ which equals L_1 on $S^3 \times \{0\}$ and L_2 on $S^3 \times \{1\}$.

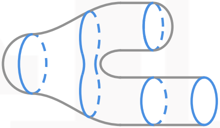


The unknot is cobordant to the two-component unlink.

Cobordism, continued

Theorem

Any cobordism may be viewed as a sequence of births, saddles, and deaths.



Corresponding surface

Concordance

Definition

A **concordance** between knots is a cobordism which is topologically homeomorphic to the cylinder $S^1 \times [0, 1]$.

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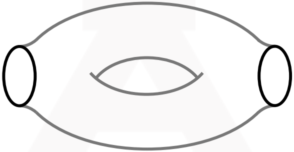

A cobordism from the unknot to itself which is **not** a concordance

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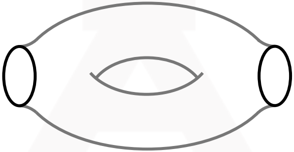

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A cobordism from the unknot to itself which **is** a concordance

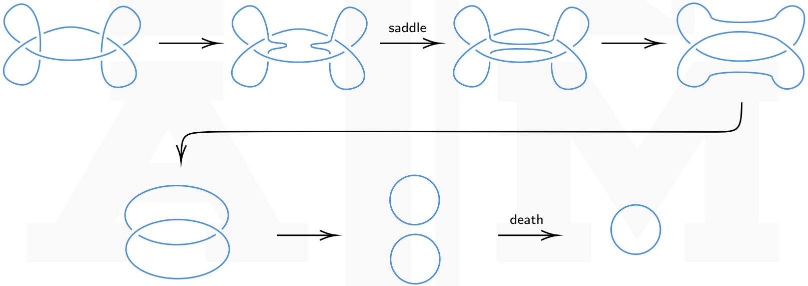
Concordance

A concordance between n-component links is a cobordism which is topologically homeomorphic to the disjoint union of n cylinders.

Slice Links

Definition

A **slice link** is an n-component link which is concordant to the n-component unlink.



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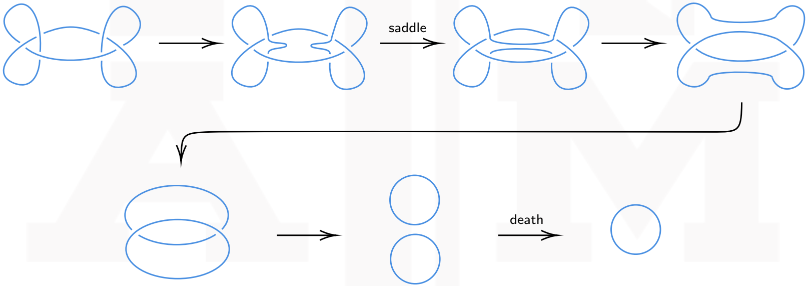
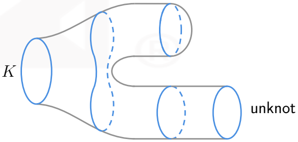


This is a saddle move followed by a death, so topologically the surface looks like this:



Ribbon Links

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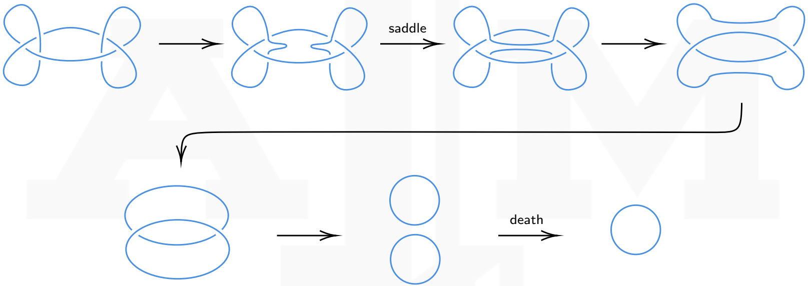
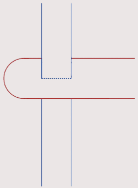


This is ribbon as well!

Ribbon Links

Theorem

An n-component link $L \subseteq S^3$ is ribbon if and only if it is the boundary of a collection of n disks immersed in S^3 such that all of its self-intersections are of the following form:



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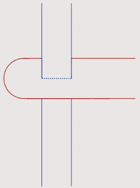

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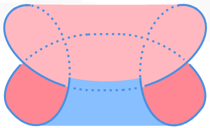
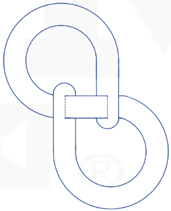


Our knot from earlier



Knot 6_1

Slice and Ribbon

All ribbon links are slice (by definition).

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Question

Are all slice links ribbon? This is known as the **slice-ribbon conjecture** and is a famous unsolved problem.

Link Invariants

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1. Linking Matrix
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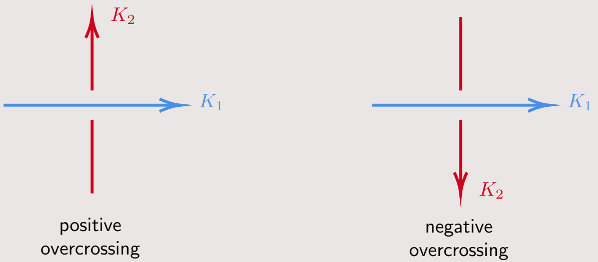
Because slice and ribbon links are concordant to the unlink, these invariants give rise to **slice obstructions** and **ribbon obstructions**.

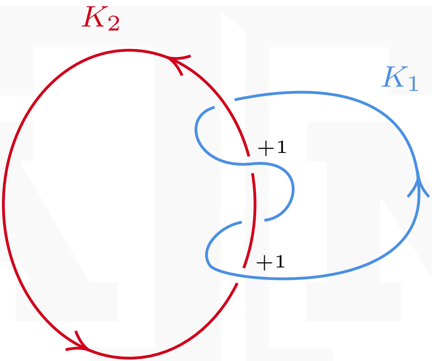
Linking Matrix

Definition

Given two oriented components of a link K_1 and K_2, their **linking number**, denoted $\mathrm{lk}(K_1, K_2)$, encodes the number of times K_1 wraps around K_2. It may be computed by fixing a link diagram and taking the number of positive minus the number of negative overcrossings of K_1 over K_2.



Linking Matrix



Linking Matrix

Definition

Given a link $L = K_1 \sqcup \cdots \sqcup K_n$, the **linking matrix** of L is the $n \times n$ matrix whose entry in the ith row and jth column equals $\mathrm{lk}(K_i, K_j)$.

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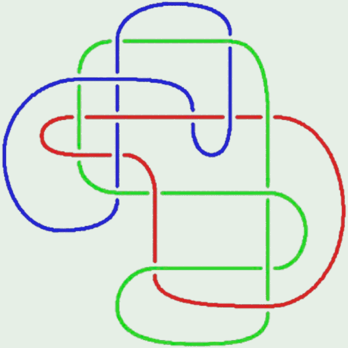

$$\text{Linking matrix} = \begin{bmatrix} 0 & -2 & 1 \\ -2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

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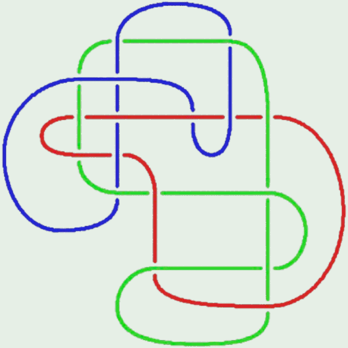

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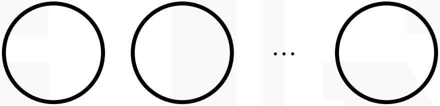
Linking Matrix

It is known that linking matrix is a concordance invariant.



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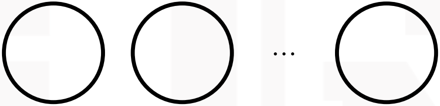

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Theorem (linking matrix slice obstruction)

Let L be a slice link. Then its linking matrix is 0.

Signature

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Given a link $L \subseteq S^3$, a **Seifert surface** for L is an orientable non-self-intersecting 2d surface in S^3 whose boundary is L.

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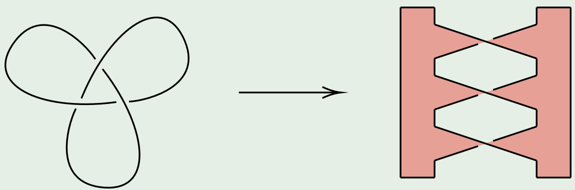
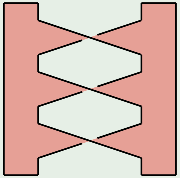


Signature

From a Seifert surface we can create something called a **Seifert Matrix**.

Example



One possible Seifert matrix $= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

This is also not unique!

Signature

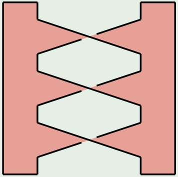
Note that a Seifert matrix, M, is always integer-valued, so $M + M^T$ is a real symmetric matrix.

Definition

Let M be a Seifert matrix for a link L. The **signature** of L, denoted $\sigma(L)$, is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the matrix $M + M^T$, counting multiplicities. That is,

$$\sigma(L) = \# \text{ positive eigenvalues } - \# \text{ negative eigenvalues}$$

Example



Our Trefoil knot from earlier had $M + M^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

So σ(Trefoil) $= 2$

Signature

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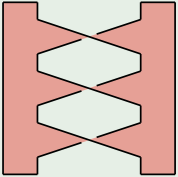

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$\sigma(L)$ is also invariant under concordance.

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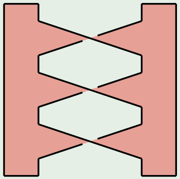

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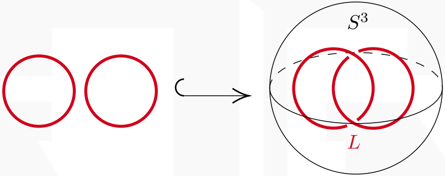
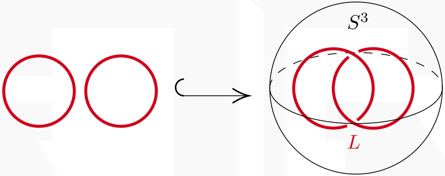
Theorem (signature slice obstruction)

If a link L is slice, then $\sigma(L) = 0$.

The Link Exterior





We call $S^3 \setminus L$ the **exterior** (or link complement) of L.
This is an isotopy invariant of links.

We want to consider the fundamental group $\pi_1(S^3 \setminus L)$.

The Alexander Module

Given an n-component link L, let $E = S^3 \setminus L$ be its exterior. Recall that the abelianization of $\pi_1(E)$ is $H_1(E)$.

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We will consider the *universal abelian cover* of E, denoted E_γ with covering map $p : E_\gamma \to E$. By definition, the deck transformation group of this covering space is $H_1(E)$.

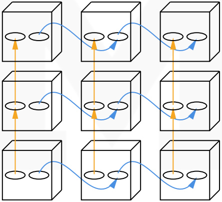
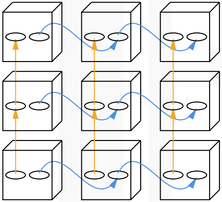


Figure: The universal abelian cover of the exterior of the 2-component unlink

Fix a point $b \in E$ and consider the relative homology group $H_1(E_\gamma, p^{-1}(b))$. The deck transformation group $H_1(E)$ acts on $H_1(E_\gamma, p^{-1}(b))$ by sending loops between sheets of E_γ.



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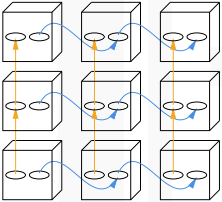


Thus, $H_1(E_\gamma, p^{-1}(b))$ is a module over the group ring $\mathbb{Z}H_1(E)$. This is called the **Alexander module** of L, denoted $A(L)$.

- Note that $\Lambda := \mathbb{Z}H_1(E)$ is isomorphic to $\mathbb{Z}[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]$.

Definition (Alexander nullity)

The **Alexander nullity** of a link L, denoted $\beta(L)$, is $\operatorname{rank} A(L) - 1$.

$\beta(L) \leq n - 1$ for any n-component link. When $\beta(L) = n - 1$ we say that it has **full Alexander nullity**.

This is is a concordance invariant of links. Since the n-component unlink has full nullity, we have

Theorem (Alexander nullity slice obstruction)

If a link L is slice, then L has full Alexander nullity.

From these invariants, we get our slice obstructions:

Slice Obstructions

Let L be an n-component slice link. Then all of the following hold:

- The linking matrix of L is 0.
- The signature of L is 0.
- L has full Alexander nullity, meaning $\beta(L) = n - 1$.

The Jones Polynomial

The Jones polynomial is a link invariant that encodes combinatorial information of a link.

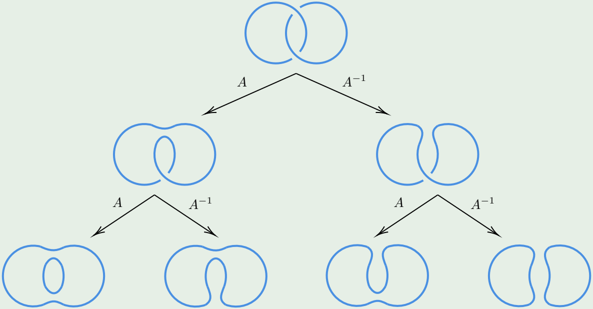
Definition

Let D be a diagram for a link L. The **Kauffman bracket** of D is a Laurent polynomial $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ which satisfies the following recursive relations:

- $\langle \bigcirc \rangle = 1$.
- $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$.
- $\langle D \sqcup \bigcirc \rangle = \langle D \rangle \cdot (-A^2 - A^{-2})$

Example

We begin with a link, D, and apply the skein relation to each crossing as follows:



We know $\langle \bigcirc \rangle = 1$ and $\langle \bigcirc^2 \rangle = (-A^2 - A^{-2})$. Thus, we get

$$\begin{aligned}\langle D \rangle &= A^2(-A^2 - A^{-2}) + 1(1) + 1(1) + A^{-2}(-A^2 - A^{-2}) \\ &= -A^4 - A^{-4}\end{aligned}$$

Jones Polynomial

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$$\mathrm{wr}(D) = \#\text{ positive crossings} - \#\text{ negative crossings}$$

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Definition

Let L be a link and D be any diagram for$L$. The **Jones polynomial** of L is given by:

$$V_L = (-A^3)^{-\mathrm{wr(D)}} \langle D \rangle$$

This is an isotopy invariant of links. Most literature substitutes $q = -A^{-2}$ so that $V_L(q) \in \mathbb{Z}[q, q^{-1}]$.

Eisermann's Condition

Definition

Given a link L, its **Jones nullity** is the multiplicity of the factor $(q + q^{-1})$ in its Jones polynomial.

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Theorem (Eisermann, 2008)

Every n-component ribbon link has Jones nullity $n - 1$. In this case, we say L has ***full Jones nullity****.*

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Question

Is Eisermann's condition a slice obstruction?

Section 3

Our Conjecture

After testing all links of up to 14 crossings, we make the following conjecture:

Conjecture

Let L be a link with $n \geq 2$ components. If L has full Alexander nullity, then

- L has linking matrix 0.
- L has even signature.
- L has full Jones nullity.

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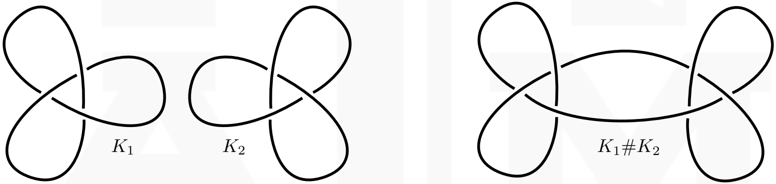
It is sufficient to check a smaller class of links called **prime links**, which we now define.

Connected Sums

Connected sum is an operation between two knots that produces a new knot, denoted $K_1 \# K_2$. It is well defined up to isotopy.



Connected sum of the trefoil with its mirror image

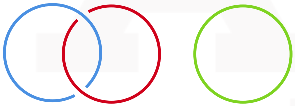
The connected sum of two links is made by performing a connected sum between one component from each link. This depends on the choice of components.

Definition

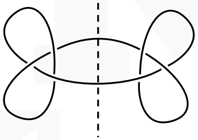
A link L is called **split** if it is a disjoint union of two links that do not share any crossings.

Definition

A link L is called **prime** if it is non-split and not a connected sum of two non-trivial links.



Split link



Non-prime link

Connected Sums

It turns out most properties behave nicely under connected sums:

- Connected sum maintains linking matrix zero
- Signature is additive. That is, $\sigma(L_1 \# L_2) = \sigma(L_1) + \sigma(L_2)$.
- The Jones polynomial is multiplicative: $V_{L_1 \# L_2}(q) = V_{L_1}(q) \cdot V_{L_2}(q)$. In particular, Jones nullity is additive.

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We are able to prove the following:

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Therefore,

Theorem

If there exists a counter-example to our conjecture, then there exists a prime counter-example.

Connected Sums

Lemma

Alexander nullity is additive under connected sum. That is, for any links L_1 and L_2 we have $\beta(L_1 \# L_2) = \beta(L_1) + \beta(L_2)$.

Proof idea:

1. Use the **Wirtinger presentation** to show that $\pi_1(S^3 \setminus L_1 \# L_2)$ is a union of presentations for $\pi_1(S^3 \setminus L_1)$ and $\pi_1(S^3 \setminus L_2)$ with one additional relation.
2. Use **Fox calculus** to compute a **presentation matrix** for $A(L_1 \# L_2)$ in terms of the presentation matrices of $A(L_1)$ and $A(L_2)$.
3. Compute the rank of the matrix to get $\beta(L_1 \# L_2)$.

Presentation Matrix

It's often easier to analyze the Alexander module through a **presentation matrix**. (Recall that $A(L)$ is a module over $\Lambda := H_1(E)$)

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A presentation matrix for $A(L)$ is a matrix P with entries in Λ for which

$$\Lambda^m \xrightarrow{P} \Lambda^n \to A(L) \to 0$$

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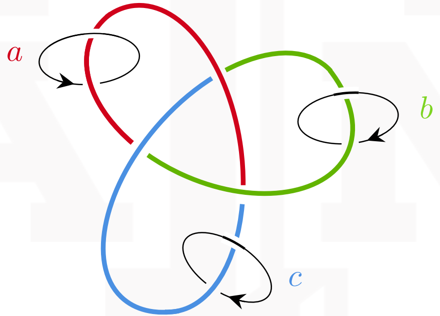
Note: the Alexander nullity $\beta(L) = \#\text{rows}\, P - \operatorname{rank} P - 1$.

This can be computed from a diagram of the link!

Wirtinger Presentation

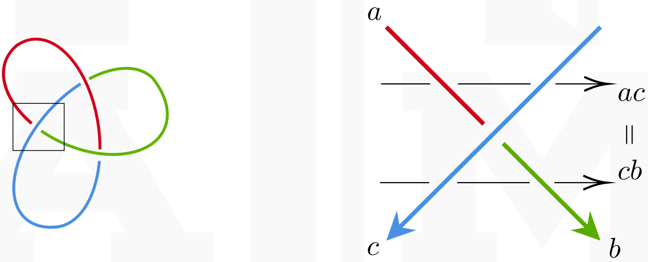
$\pi_1(S^3 \setminus L)$ can be computed from a link diagram of L.

Generators correspond to loops around arcs on the diagram:



Wirtinger Presentation

Relations correspond to crossings:



Note that the relation can also be written as $a = cbc^{-1}$.

Wirtinger Presentation

Together, these give us a presentation of $\pi_1(S^3 \setminus L)$ called the **Wirtinger presentation**.

The presentation of the trefoil would be:

$$\pi_1(S^3 \setminus \text{trefoil}) = \langle a, b, c \mid a = cbc^{-1}, b = aca^{-1}, c = bab^{-1} \rangle.$$

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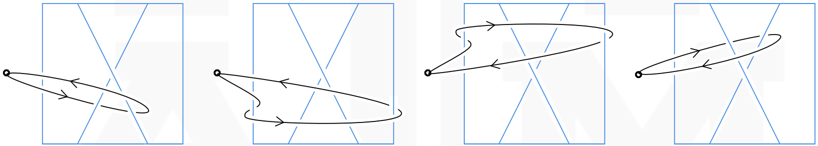
Note the last relation can be derived from the previous two:

$$\begin{aligned} b &= aca^{-1} \\ &= (cbc^{-1})ca^{-1} \\ &= cba^{-1} \\ \implies c &= bab^{-1} \end{aligned}$$

In fact, this is true in general: In a Wirtinger presentation, each relation is dependent on all of the other ones.



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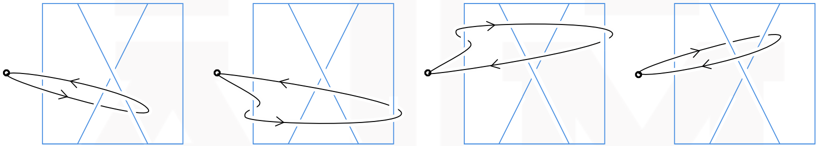


Every non-split, non-trivial link diagram has the same number of arcs as crossings, so

$$\# \text{ relations} = \# \text{ generators} - 1$$

Fox derivative

We now need a way to use $\pi_1(E)$ to get information about the universal abelian cover.

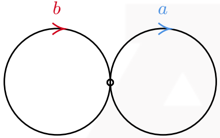
Definition

Let G be a free group with generators $x_1, \ldots, x_n$. We define the **Fox derivative** with respect to x_i to be the map $\frac{\partial}{\partial x_i} : G \to \mathbb{Z}G$ given by the following rules:

- $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$
- $\frac{\partial}{\partial x_i}(uv) = \frac{\partial}{\partial x_i}(u) + u\frac{\partial}{\partial x_i}(v)$ for any $u, v \in G$

Fox derivative intuition

The free group on two generators $\langle a, b \rangle$ is the fundamental group of a bouquet of two circles. Call this space E.



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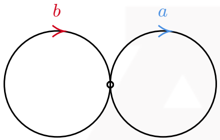
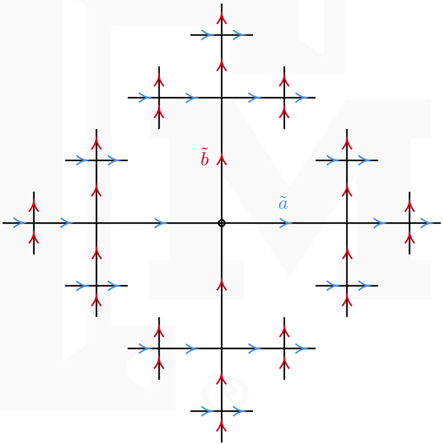


Consider the universal cover of this space (call it $\widetilde{E}$) and fix lifts $\widetilde{a}$ and $\widetilde{b}$ with the same base point:



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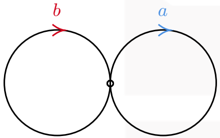
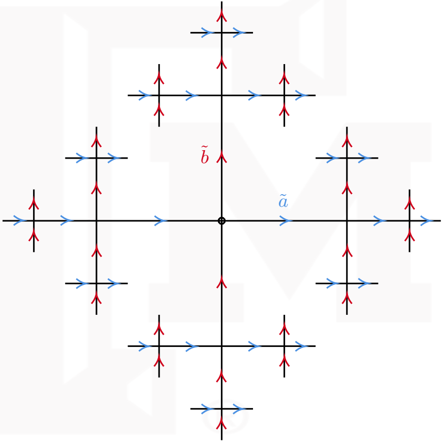


The deck group of $\widetilde{E}$ may be identified with $\pi_1(E) = \langle a, b \rangle$.



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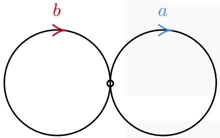
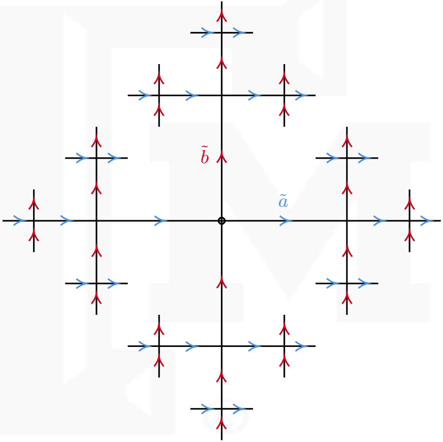


The deck group of $\widetilde{E}$ may be identified with $\pi_1(E) = \langle a, b \rangle$.

We may view the paths $\widetilde{a}$ and $\widetilde{b}$ as 1-cells and let a and b act on $\widetilde{a}$ and $\widetilde{b}$ as deck transformations.



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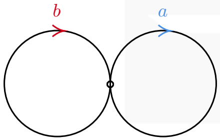
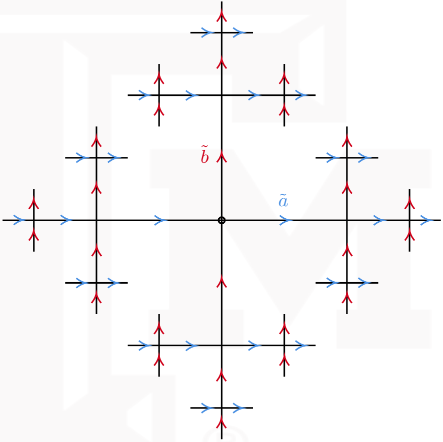


The idea of the Fox derivative is that given a loop $u \in \pi_1(E) = \langle a, b \rangle$, the lift of u to $\widetilde{E}$ is equal to

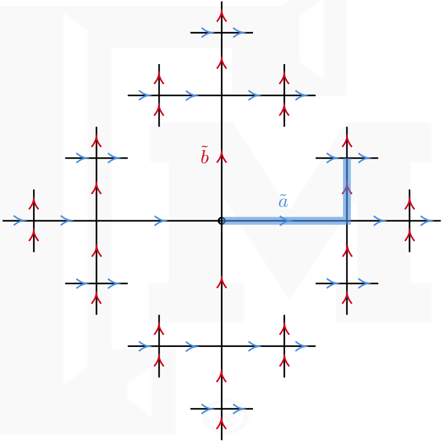
$$\frac{\partial u}{\partial a}\widetilde{a} + \frac{\partial u}{\partial b}\widetilde{b},$$

viewed as a 1-chain in $\widetilde{E}$.



Fox derivative intuition

For example, consider the lift of the word ab. This corresponds to the chain in $\widetilde{E}$ highlighted in blue:



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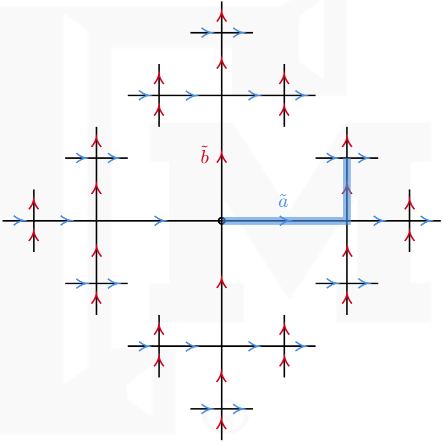

From the picture we can see that the lift is $\widetilde{a} + a\widetilde{b}$. This corresponds to the rule

$$\frac{\partial}{\partial a}(uv) = \frac{\partial}{\partial a}(u) + u\frac{\partial}{\partial a}(v).$$

Fox derivative

The Fox derivative gives us information about the universal cover. We use the abelianization map $\gamma : \pi_1(S^3 \setminus L) \to H_1(S^3 \setminus L)$ to extract information about the universal abelian cover. Let G be the free group on the generators of $\pi_1(S^3 \setminus L)$, then we have maps

$$G \xrightarrow{\partial/\partial x} \mathbb{Z}G \xrightarrow{\gamma_*} \Lambda$$

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Take the Fox derivative of all relations of $\pi_1(S^3 \setminus L)$ with respect to all generators to get:

$$P = \begin{pmatrix} \gamma_*(\partial r_1/\partial x_1) & \cdots & \gamma_*(\partial r_{p-1}/\partial x_1) \\ \vdots & \ddots & \vdots \\ \gamma_*(\partial r_1/\partial x_p) & \cdots & \gamma_*(\partial r_{p-1}/\partial x_p) \end{pmatrix}$$

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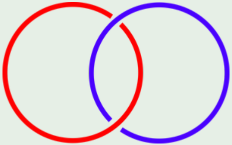
Theorem (Kawauchi, 7.1.5)

P defines a map $\Lambda^{p-1} \to \Lambda^p$ such that $\operatorname{coker} P \cong A(L)$. In other words, P is a presentation matrix for $A(L)$.

Fox Derivative

Example



Fundamental and homology groups:

$$\pi_1(E) = \langle a, b \mid aba^{-1}b^{-1} \rangle \qquad H_1(E) = \langle t_1 \rangle \times \langle t_2 \rangle$$

Fox derivatives:

$$\frac{\partial}{\partial a}(aba^{-1}b^{-1}) = 1 - aba^{-1}$$
$$\frac{\partial}{\partial b}(aba^{-1}b^{-1}) = a - aba^{-1}b^{-1}$$

Abelianization:

$$\gamma_*(1 - aba^{-1}) = 1 - t_2, \qquad \gamma_*(a - aba^{-1}b^{-1}) = t_1 - 1.$$

Presentation matrix:

$$P = \begin{bmatrix} 1 - t_2 \\ t_1 - 1 \end{bmatrix}, \qquad \beta(L) = 2 - 1 - 1 = 0.$$

Lemma

Alexander nullity is additive under connected sum. That is, for any links L_1 and L_2 we have $\beta(L_1 \# L_2) = \beta(L_1) + \beta(L_2)$.

Using these tools, we were able to show that the presentation matrix for $A(L_1 \# L_2)$ is of the form:

$$P_{\#} = \begin{bmatrix} P_1 & 0 & e_1 \\ 0 & P_2 & -f_1 \end{bmatrix}$$

from which additivity of the nullity followed.

Some results

We give a result on the Jones nullity of 2-component slice links.

Theorem

Let L be a 2-component link with even signature. Then $q + q^{-1}$ divides $V_L(q)$. In other words, L has full Jones nullity.

Corollary

Every 2-component slice link has full Jones nullity

Conjectures, continued

We continue to attempt to expand our previous theorem. Proving either of the following conjectures will show that all slice links have full Jones nullity.

Conjecture

Alexander nullity is a lower bound for Jones nullity.

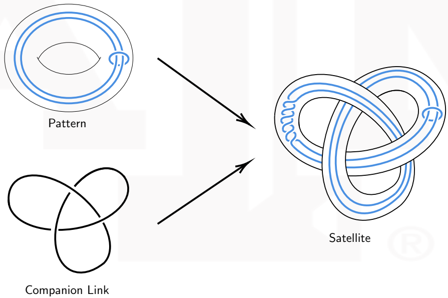
Conjecture

Jones nullity is a concordance invariant.

Definition

Let K be a knot, and let P be a link embedded in a solid torus. We define a **satellite** of K with pattern P to be the image of P under an embedding of the solid torus onto a tubular neighborhood of K.

The embedding in the above definition is required to "have no twists" in the sense that it preserves linking number.



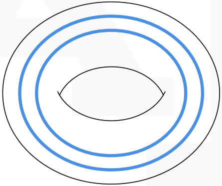
Satellite links

We can prove our conjecture is true for a specific class of satellite links:

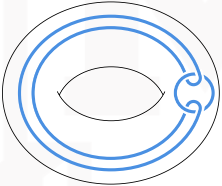
Theorem

Let L be a satellite link whose companion is a knot and whose pattern is an n-component unlink. Then L satisfies the following:

- *L has full Alexander nullity*
- *L has even signature*



Cable



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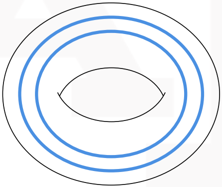

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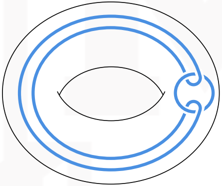
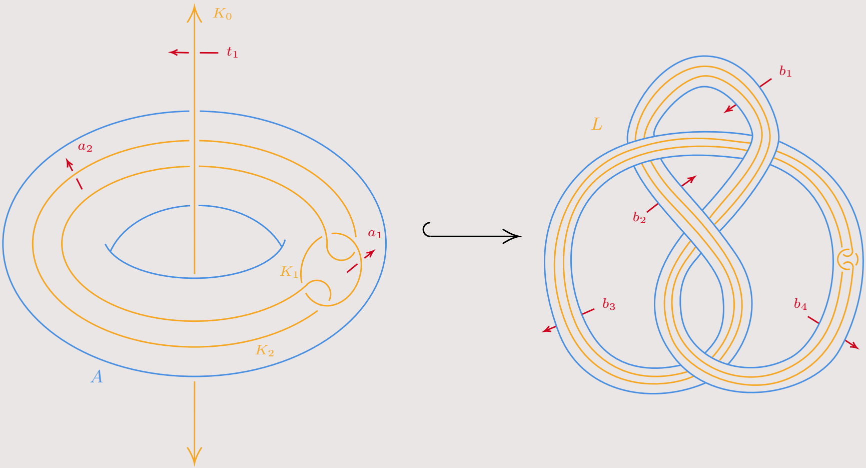


Bing double

Recall from earlier that a 2-component link with even signature has full Jones nullity.

Proof outline.

Full Alexander nullity:



□

Proof outline.

Full Alexander nullity:

$$
\begin{array}{c}
\\
b_1\\ \vdots \\ b_p \\ t_1 \\ \vdots \\ t_d \\ a_1 \\ \vdots \\ a_n
\end{array}
\begin{array}{cccccccc}
r_1 & \cdots & r_{p-1} & s_1 & \cdots & s_{n+d-1} & [m] & [\ell] \\
\gamma(\frac{\partial r_1}{\partial b_1}) & \cdots & \gamma(\frac{\partial r_{p-1}}{\partial b_1}) & 0 & \cdots & 0 & \gamma(\frac{\partial([m])}{\partial b_1}) & \gamma(\frac{\partial[\ell]}{\partial b_1}) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
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0 & \cdots & 0 & \gamma(\frac{\partial s_1}{\partial t_1}) & \cdots & \gamma(\frac{\partial s_{n+d-1}}{\partial t_1}) & \gamma(\frac{\partial[m]}{\partial t_1}) & \gamma(\frac{\partial[\ell]}{\partial t_1}) \\
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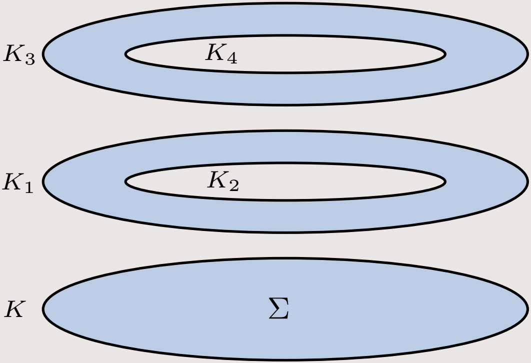
0 & \cdots & 0 & \gamma(\frac{\partial s_1}{\partial t_d}) & \cdots & \gamma(\frac{\partial s_{n+d-1}}{\partial t_d}) & \gamma(\frac{\partial[m]}{\partial t_d}) & \gamma(\frac{\partial[\ell]}{\partial t_1}) \\
0 & \cdots & 0 & 0 & \cdots & 0 & \gamma(\frac{\partial[m]}{\partial a_1}) & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \gamma(\frac{\partial[m]}{\partial a_n}) & 0
\end{array}.
$$

□

Proof outline.

Even signature:

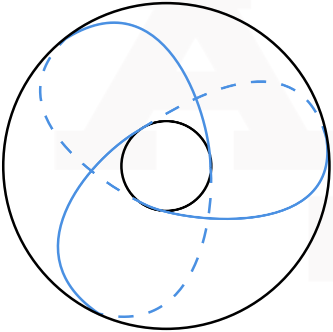


□

Torus links

Definition (Torus knot)

Let $K_1 \sqcup K_2$ be the Hopf link, and let $N(K_1)$ be a tubular neighborhood of K_1 disjoint from K_2. For coprime integers p and q, we define the (p, q) torus knot, denoted $T(p, q)$, to be the knot lying on the boundary of $N(K_1)$ (a torus) for which $(T(p, q), K_1) = q$ and $(T(p, q), K_2) = p$.



$T(2, 3)$

Torus links

Definition (Torus link)

Let p and q be coprime integers, and let n be a positive integer. Then we define the (p, q) torus n-link, denoted $T(np, nq)$, to be the link consisting of n parallel copies of $T(p, q)$ lying on a torus embedded in S^3.

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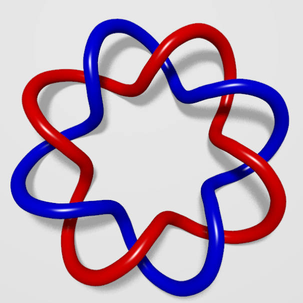
Recall this diagram from the beginning:



$T(2, 8)$

Torus links

Nontrivial torus links have nonzero linking matrix, so our conjecture would have us believe that they don't have full Alexander nullity. We can prove this:

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Let p and q be nonzero coprime integers, and let $n \geq 2$. Then the torus link $T(np, nq)$ has Alexander nullity 0.

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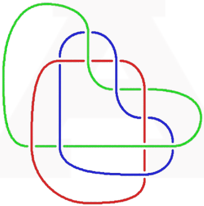
Proof outline.

$$\begin{bmatrix} \frac{t_n^{pq}-1}{t_n^q-1} & \frac{t_n^{pq}-1}{t_n^q-1} & \frac{t_n^{pq}-1}{t_n^q-1} & \cdots & \frac{t_n^{pq}-1}{t_n^q-1} \\ -\frac{t_n^{pq}-1}{t_n^p-1} & -t_1\frac{t_n^{pq}-1}{t_n^p-1} & -t_2\frac{t_n^{pq}-1}{t_n^p-1} & \cdots & -t_{n-1}\frac{t_n^{pq}-1}{t_n^p-1} \\ 0 & t_n^{pq}-1 & 0 & \cdots & 0 \\ 0 & 0 & t_n^{pq}-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_n^{pq}-1 \end{bmatrix}$$

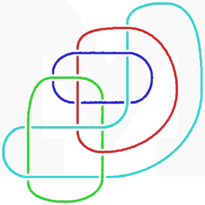
□

Eisermann asked whether the Jones nullity of every link is equal to its Murasugi nullity, $\mathrm{null}(L)$. Using the SnapPy Python library, we have found the following minimal counterexamples:

- The 3-component prime link “L12n1998” has Jones nullity 1 and Murasugi nullity 2. (it also has Alexander nullity 0).
- The 4-component prime link “L14n63006” has Jones nullity 2 and Murasugi nullity 1. (it also has Alexander nullity 0).



L12n1998



L14n63006

Conclusion

1. Conjectured that even signature and full Jones nullity follow from full Alexander nullity
2. Proved the sufficiency of checking prime links
3. Proved part of this conjecture in specific cases:
    - 2-component links
    - Unlinked satellites
    - Torus links

Conclusion

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Future Work:

- Prove full Jones nullity for unlinked satellites
- Expand the conjecture to more classes of links

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Thank You!