

Random geometry and Percolation:

- Percolation theory on \mathbb{Z}^d and general graphs.
- Critical behavior
- Modern tools such as renormalization, stochastic dimension.
- Random walk on random graphs.
- Correlated percolation models: Random interlacements and Gaussian free field.

Books: - Grimmett - percolation 2nd edition.

- online {
- Benjamini - Coarse Geometry and Randomness
 - Dreieritz, Rath, Sapozhnikov - An introduction to Random Interlacements
 - papers.

Credits - Exercises (every two weeks).

- Can choose to present a one hour lecture (not easy).

Percolation on graphs: Given a graph $G = (V, E)$

Percolation on G is a random diluting of the graph.

Let $\Omega = \{0, 1\}^E$, \mathcal{F} - cylinder σ -algebra (events which depend on a finite # of edges), $p \in [0, 1]$, $\mu_p(1) = p$, $\mu_p(0) = 1 - p$

$\mathbb{P}_p = \prod_{e \in E} \mu_p$ (product measure). Bond percolation on G_x is $(\Omega, \mathcal{F}, \mathbb{P}_p)$ with parameter p

can be realized by random variables. π_e - projection to edge e .

Then $\{\pi_e\}_{e \in E}$ are i.i.d $\sim \mu_p$.

Percolation is the simplest model which exhibits critical phenomena: there is a natural parameter in the model which admits a drastic behavior change.

What are the typical questions:

Definition: The component of a vertex $v \in V$ is the set of all vertices that can be reached from v using a path of open edges ($\kappa_e = 1$),

$$C(v) = \left\{ w \in V : \exists n \in \mathbb{N}, v_0, v_1, \dots, v_n \in V, v_0 = v, v_n = w, (v_i, v_{i+1}) \in E \right. \\ \left. \prod_{v_i, v_{i+1}} \kappa_{e} = 1 \quad \forall 0 \leq i \leq n-1 \right\},$$

Q₁: Are there infinite components?

Q₂: Does this depend on p ?

Q₃: Can one compute the critical value? (still open in \mathbb{Z}^3).

Q₄: How many infinite components are there?

Q₅: What is the size of finite open components?

Q₆: What is the typical graph distance between two vertices in an open component.

The main examples to keep in mind $G = \mathbb{Z}^d$ or \mathbb{T}_d ^{d -regular tree,}

Theorem: Let $G = \mathbb{Z}^d$, then

1. if $d \geq 1$ and $p > 3/4$ then 0 is connected to \mathcal{O} with positive probability.

2. if $d \geq 2$ and $p < 1/3$ then 0 is not connected to \mathcal{O} w.p 1.

Remark: 1. enough to prove first part for $d=2$. (subgraph).

2. theorem suggests existence of a phase transition for existence of inf cluster.

Definition: For a graph G and a fixed root g ,

$$\rho_c(G, g) = \max \{ 0 \leq p \leq 1 : \rho_p(g \leftrightarrow \infty) > 0 \}$$

Ex. If G is a connected graph then $\rho_c(G, g)$ does not depend on g .

Corollary: $\frac{1}{3} < \rho_c(\mathbb{Z}^2) < \frac{3}{4}$ (non trivial).

Ex! Is there a connected graph G with $\rho_c(G) = 1$? (\mathbb{Z})
what about $\mathbb{Z} \times \mathbb{Z}_2$ (dense Cartesian product).

Thm: Let G be a connected graph with $\deg(v) \leq d \quad \forall v \in V$ then

$$P_c(G) \geq \frac{1}{d-1}$$

Proof: If $f \leftrightarrow \infty$ then there is an open simple (self avoiding)

Path from f to ∞ . The # of self avoiding paths of

length n starting at f is at most $d(d-1)^{n-1}$ (Right for what graph?)
The prob for such a path to be open is p^n

$$\Rightarrow P_p(f \leftrightarrow \infty) \leq P_p(\exists \gamma^{\text{open}}, \text{ self avoiding, } |\gamma| = n, \text{ from } f) =$$

$$\leq P\left(\bigcup_{\substack{\gamma \text{ self avoiding} \\ |\gamma| \geq n}} \gamma \text{ is open}\right) \leq \sum P(\cdot) = d(d-1)^{n-1} p^n \xrightarrow[n \rightarrow \infty]{\text{if } p < \frac{1}{d-1}} 0 \quad \square$$

Corollary: 1. $\rho_c(\mathbb{T}_d) \geq \frac{1}{d-1}$ (actually $= \frac{1}{d-1}$)

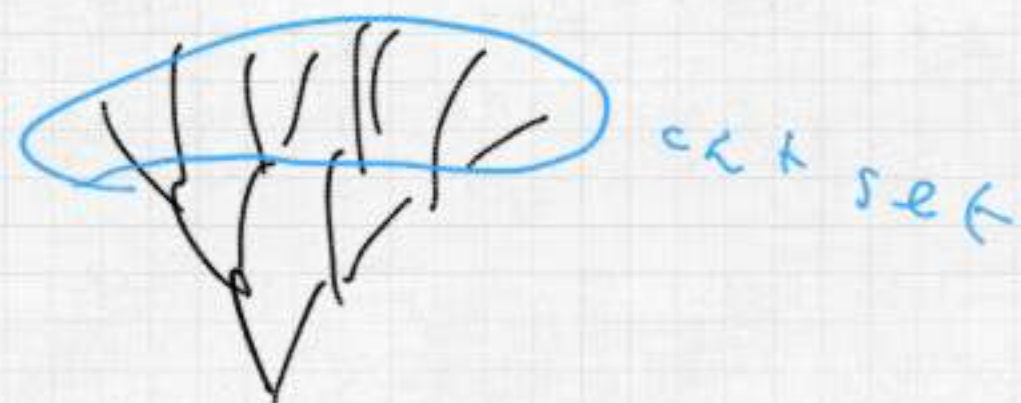
2. $\rho_c(\mathbb{Z}^d) \geq \frac{1}{2d-1} \Rightarrow \rho_c(\mathbb{Z}^2) \geq \frac{1}{3}$

Definition: For G with a root s , we say that a set of edges is a cut set if every path from s to ∞ must cross an edge from the set.

Example: $G = \mathbb{Z}$ then $\{(s, -6), (7, 8)\}$ is a cut set

$\{(3, 4), (100, 101)\}$ is not.

$G = \mathbb{T}_2$



Definition

A cut set is said to be a minimal cut set (MCS) if any $T \subsetneq S$ is not a cut set.

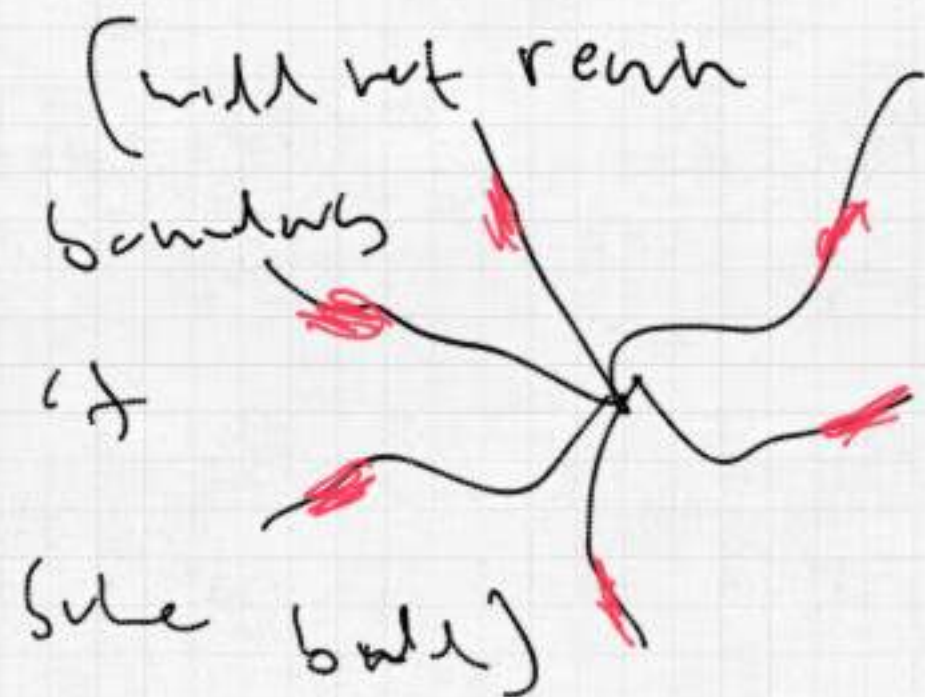
Theorem Let G be a connected infinite graph with root g . Is $\exists C > 0$ s.t. $\forall n \geq 0$ $\# \{ \text{MCS of size } n \} < C^n$ for $P_c(G) < 1$. (What about $G = \mathbb{Z}$? $n \geq 2$ is get inf)

Proof If $g \leftrightarrow \emptyset$ then there is a MCS which is all closed.

If S is a cutset of size n ,

$$P(S \text{ closed}) = (1-p)^n$$

$$\Rightarrow P(\exists \text{ closed MCS of size } n) \leq (C(1-p))^n$$



$$\Rightarrow \mathbb{P}_p(\exists \text{ closed MCS}) \leq \sum_{h=1}^{\infty} (C(1-p))^h \xrightarrow{p \rightarrow 1} 0$$

$\Rightarrow \exists p_0 = 1 \quad \forall p > p_0 \quad \exists \delta > 0$ with positive probability,

Exponential Intersection Tail; (Kesten)

Definition: A rooted graph (G, ρ) admits exponential intersection tail (EIT) $\Leftrightarrow \exists$ a measure μ on paths in G , supported only on infinite paths from ρ , with the following property: $\exists \theta < \theta < 1$

$$\forall k \quad \sum_{\mu \times \mu} (|\delta_1 \cap \delta_2| > k) < \theta^k$$

Prop 1: The binary tree has EIT property.

Proof: μ uniform over monotone paths (SRW-directed).

$$P(|a_{t+h}| > h) = \frac{1}{2^n} \quad \text{QED}$$

Prop 1: \mathbb{Z}^2 does not have EIT.

Proof: Let μ be a measure over paths from 0.

$\tilde{\mu}_n(x)$ = prob first hit on $\partial[-n, n]$ is in X .

$$P_{\mu \times \mu}(\text{paths intersect in } \partial[-n, n]^2) \geq \sum_{X \in \partial[-n, n]^2} \tilde{\mu}_n(x)^2 \stackrel{\text{Cauchy-Schwarz}}{\geq} \frac{(\sum_{X \in \partial[-n, n]^2} \tilde{\mu}_n(x))^2}{(\sum_{X \in \partial[-n, n]^2} 1)^{1/2}} = \frac{1}{4n}$$

$$E_{m \times m} [\# \text{ of intersections}] = \sum_{k=1}^d E_{m \times m} [\# \text{ of intersections on the } S_k \text{ plane of size } k] > \sum_{k=1}^d \frac{c}{k} \approx \infty$$

> 1. if \neq intersection.

but EIT \Rightarrow finite



Open problems Does Loop erased rw on \mathbb{Z}^d admit EIT?

prop! m uniform on rotation points admit EIT in \mathbb{Z}^d when $d \geq 4$.

Theorem If G admits FIT with parameter θ then
 $\rho_c(G) \leq \theta$.

Proof Let μ be a measure satisfying FIT- θ .

Assume

$$Z_n = \int \rho^{-n} \mathbb{1}_{\{\delta[0, n] \text{ is open}\}} d\mu$$

with ρ^{-n} this is $\mu(\text{points stay in the open cluster } \rightarrow 0 \text{ for } n \text{ steps})$.

\Rightarrow need to show $z > 0$ s.t. $\mathbb{P}_\rho(z_n > 0) > c$ b.w.v.

by fusion:
$$\mathbb{E}[z_n] = \int_{\mathcal{Z}} \rho^{-n} \cdot 1 \cdot \rho^n d\mu(x) = 1$$

by Paley Zygmund inequality. (Cauchy-Schwarz),

$$\mathbb{P}_\rho(z_n > 0) \geq \frac{\mathbb{E}_\rho[z_n]^2}{\mathbb{E}_\rho[z_n^2]} > 1$$

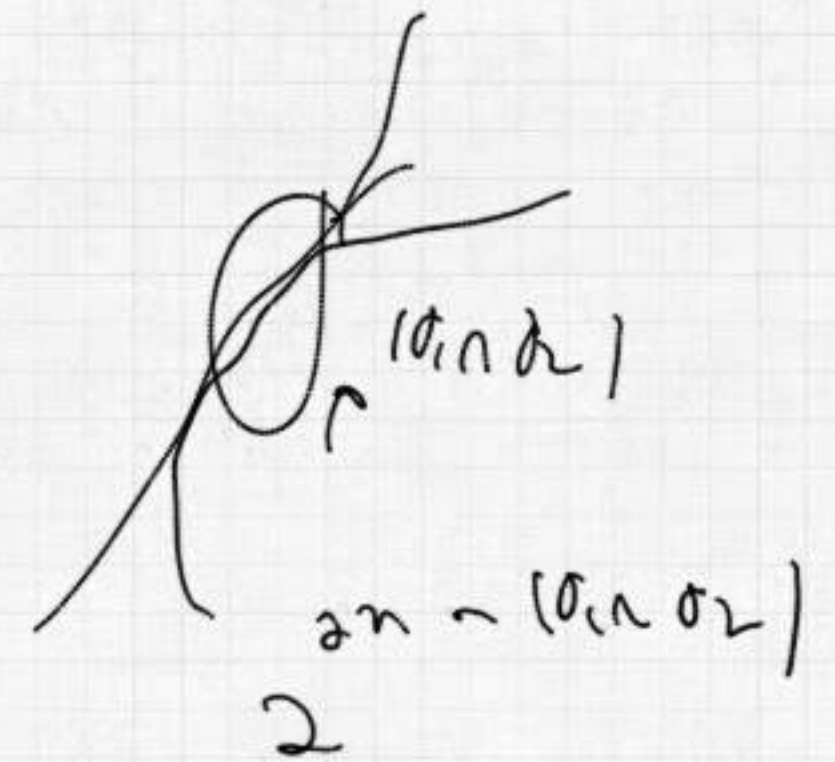
\Rightarrow we need a uniform upper bound on $\mathbb{E}_\rho[z_n^2]$.

$$\mathbb{E}_\rho[z_n^2] = \mathbb{E}_\rho \left[\int_{\sigma_1}^{\sigma_2} \rho^{-2n} \mathbb{1}_{\rho^{\sigma_1} \text{ and } \sigma_2 \text{ are open}} d_\mu(\sigma_1) d_\mu(\sigma_2) \right]$$

$$\leq \int_{\sigma_1} \int_{\sigma_2} \rho^{-|\sigma_1 \cap \sigma_2|} d\mu(\sigma_1) d\mu(\sigma_2) \stackrel{\text{F.T.}}{\leq} \sum_{i=0}^{\infty} \left(\frac{\rho}{\lambda}\right)^i$$

if $\rho > 0$ sum is finite

Ex: ρ_i (d-regular tree) = $\frac{1}{d}$.



Ergodic theory: Given a measure space $(\mathcal{N}, \mathcal{F}, \mu)$

and a measurable map $T: \mathcal{N} \rightarrow \mathcal{N}$. T is measure preserving if

$$\forall A \in \mathcal{F} \quad \mu(A) = \mu(T^{-1}(A)).$$

Given $(\mathcal{N}, \mathcal{F}, \mu, T)$ $E \in \mathcal{F}$ is called invariant if $T^{-1}(E) = E$.

Definition: A measure preserving space $(\mathcal{N}, \mathcal{F}, \mu, T)$ (complete) is called ergodic if $\forall A \in \mathcal{F}$ s.t. $T^{-1}A = A$, $\mu(A) \in \{0, 1\}$.

Example: Percolation on \mathbb{Z}^d $T = \theta_1$ - shift in e_1 direction

Proof: A, B cylinder events $\mu(A \cap T^{-n}B) \rightarrow \mu(A)\mu(B)$.

Remark on general
events not cylinders

∞

no strong mixing for σ -finite measures

Definition: A probability preserving transformation (X, \mathcal{B}, μ, T)

is called strong mixing, if for all $E, F \in \mathcal{B}$, $\mu(E \cap T^{-n}F) \xrightarrow{n \rightarrow \infty} \mu(E)\mu(F)$

Prop: strong mixing \Rightarrow ergodicity.

Proof: If $T^{-1}(E) = E$ then $\mu(E) = \mu(E \cap T^{-n}(E)) \xrightarrow{n \rightarrow \infty} \mu(E)^2$

$\Rightarrow \mu(E) \in \{0, 1\}$. \square

Prop: (X, \mathcal{B}, μ, T) is strongly mixing iff $\forall f, g \in L^2$

$\int f g \circ T^n d\mu \xrightarrow{n \rightarrow \infty} \int f d\mu \int g d\mu$ equivalently $\text{Cor}(f, g \circ T^n) \xrightarrow{n \rightarrow \infty} 0$.

Corollary: L^2 .

Continue proof (per its ergodic): f, g indicators of cylinders

$\text{cov}(f, g \circ T^k) \xrightarrow[k \rightarrow \infty]{} 0$ - Every L^2 -function can be approximated in L^2 by a finite linear combination of indicators of cylinders, s.t. $\|f - f_\varepsilon\|_2 < \varepsilon, \|g - g_\varepsilon\|_2 < \varepsilon$.

$$|\text{cov}(f, g \circ T^k)| \leq |\text{cov}(f - f_\varepsilon, g \circ T^k)| + |\text{cov}(f_\varepsilon, g_\varepsilon \circ T^k)| \\ + |\text{cov}(f_\varepsilon, (g - g_\varepsilon) \circ T^k)|$$

$$|\text{cov}(f, g)| \leq \|f - f_\varepsilon\|_2 \|g - g_\varepsilon\|_2 \leq (\|Sf\|_2 + \|Sf_1\|_2) (\|g_\varepsilon\|_2 + \|g_1\|_2) \leq 2\|Sf\|_2 \|g\|_2$$

$$\leq 4\varepsilon \|g\|_2 + o(\varepsilon) + 4(\|Sf\|_2 + \varepsilon) \varepsilon \quad \text{as } k \rightarrow \infty$$



Theorem Tempelman: Let $t_1, \dots, t_d \in \mathbb{Q}_{>0}$, and

$\{I_r\}_{r \geq 1}$ an increasing sequence of boxes which tends to \mathbb{Z}^d .

For $f \in L_1$, then

$$\frac{1}{|I_r|} \sum_{x \in I_r} f(x) \xrightarrow{\text{a.s.}} \mathbb{E}[f \mid \text{Inv}(t_1) \cap \dots \cap \text{Inv}(t_d)] \text{ a.s.}$$

Uniqueness of the inf cluster^a

For \mathbb{T}_2 we saw that $\rho_c = \frac{1}{2}$. Let $\frac{1}{2} < \rho < 1$

How many inf comp are there?



inf long cluster
edges are
inf open.
(interchanging).

$$\rho(\infty \text{ inf comp}) = 1.$$

Uniqueness in \mathbb{Z}^d : (on any amenable transitive graph).

Define $\Theta_d^{(\rho)} = \mathbb{P}_\rho(|C(\omega)| = \infty)$.

Theorem 1 $\lambda > 0, \theta(\tau) > 0$ then $\mathbb{P}_\rho(\exists \text{ a unique int cluster}) = 1$

Proof 1 $\forall k \in \{0, 1, 2, \dots, \infty\}$ let $E_k = \{\# \text{ of int clusters is } k\}$,

E_k is inv event $\Rightarrow \mathbb{P}_\rho(E_k) \in [0, 1] \forall k$. Since E_k are
disjoint and cover the space $\exists k$ s.t. $\mathbb{P}_\rho(E_k) = 1$,

note $\mathbb{P}_\rho(\exists \text{ int cluster}) = 1$.

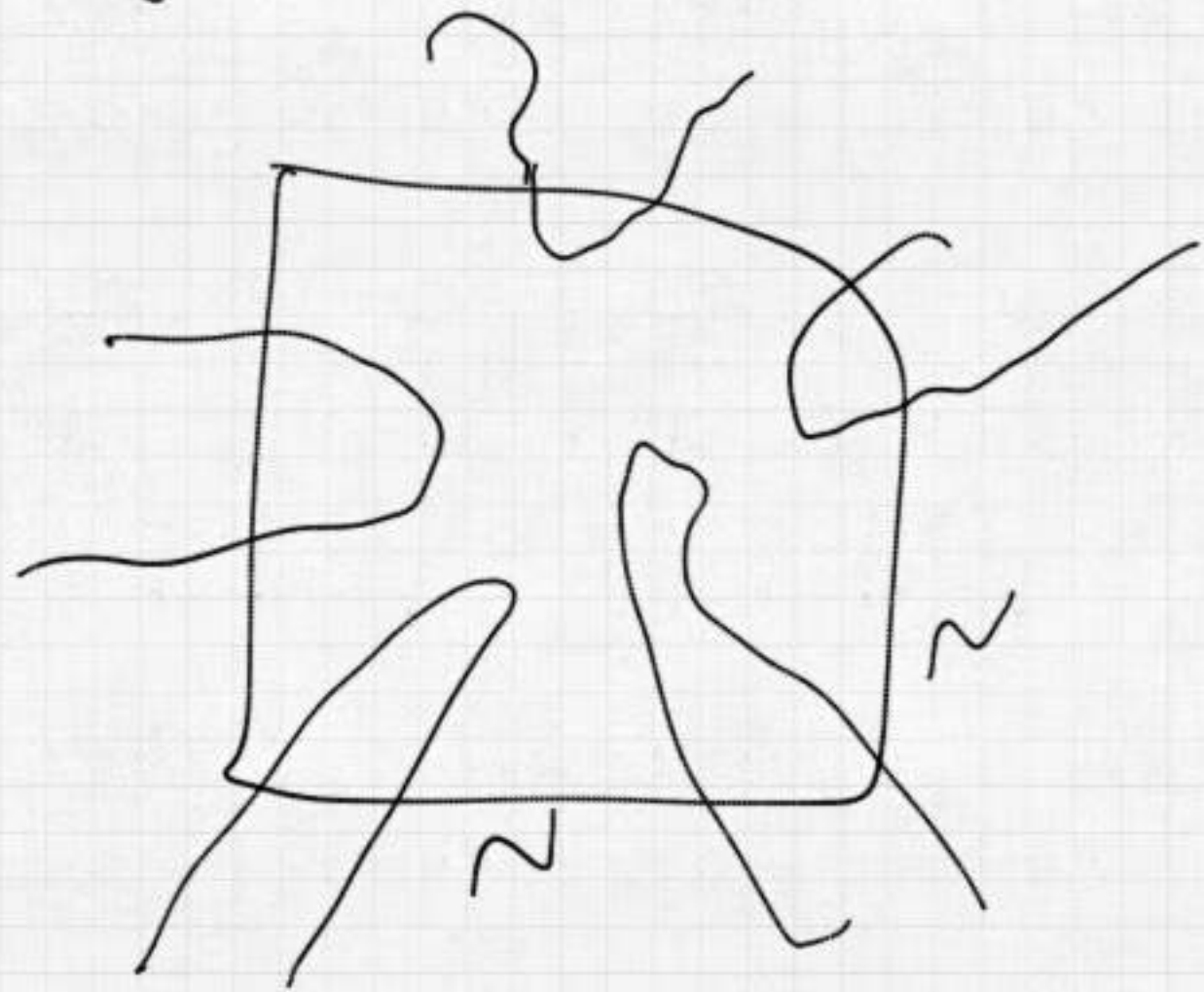
Lemma 1 $\forall k \neq 1$ finite $\mathbb{P}_\rho(E_k) = 0$.

Proof 1 let $k \neq 1$ assume $\mathbb{P}_\rho(E_k) = 1$. Define $F_n = \{\exists \text{ int clusters and each intersects } [n, n+1]^d\}$

then $F_1 \subseteq F_2 \subseteq \dots$ and $\bigcup_i F_i = E_S \Rightarrow P_p(F_i) \xrightarrow{\text{def}} 1$.

Choose N st $P_p(F_N) > 0$.

Let $\tilde{F}_N = \{ \text{5 disjoint red clusters} \}$
 outside $(-N, N)^d$ touching
 $\partial(-N, N)^d$.



$F_N \subseteq \tilde{F}_N \Rightarrow P_p(F_N) > 0$. \tilde{F}_N is not of the configuration
 in $(-N, N)^d$, (let G be the event all edges in D are
 open, $P(\tilde{F}_N \cap G) = P(\tilde{F}_N) \cdot P(G) > 0$ but $\tilde{F}_N \cap G \subseteq E_1$
 $\Rightarrow P_p(E_1) > 0$!

We are left with finding out ∞ many infinite clusters.
(Barton Keane).

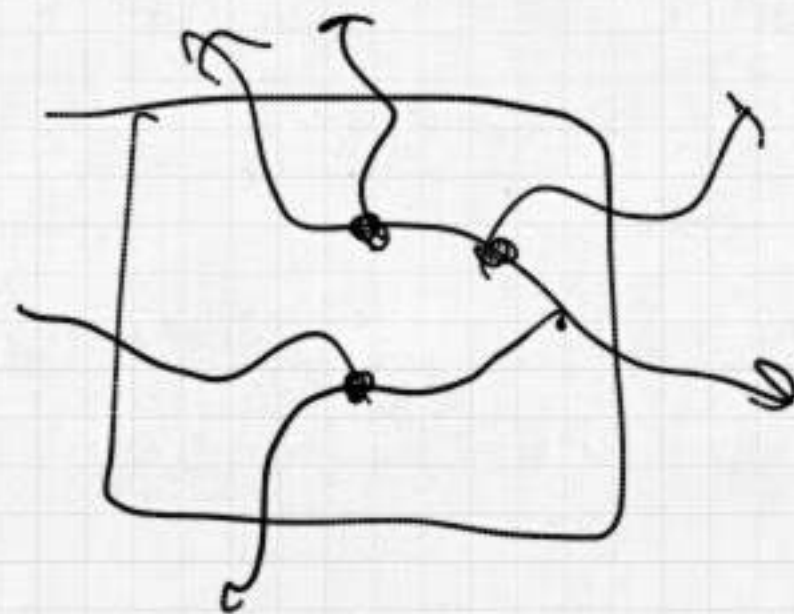
Let $Q = \#$ of its clusters, assume $P_p(Q = \emptyset) = 1$.

call $z \in \mathbb{Z}^d$ a fork point if

1. $|C(z)| = \infty$.

2. $C(z) \setminus \{z\}$ has to finite components and exactly 3

infinite components.

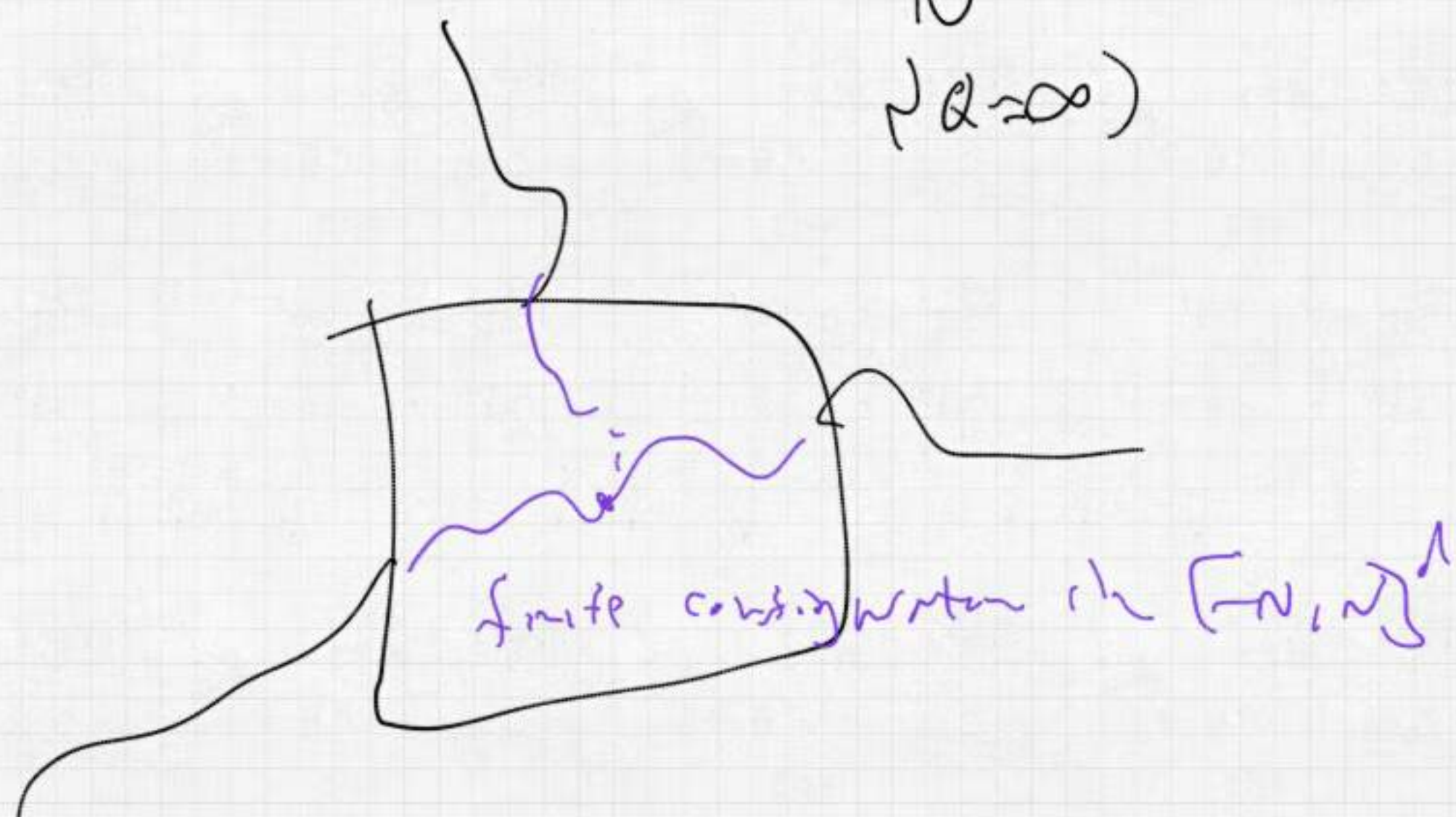


Lemma: If $\mathbb{P}_p(\mathcal{Q} = \infty) > 1$ then $\mathbb{P}_p(0 \text{ is a fork}) > 0$.

Proof: let $F_n = \{ \text{at least 3 int clusters intersect } [-n, n]^d \}$.

Since $\bigcup_i F_i = \{ \mathcal{Q} \geq 3 \}$ we have $\mathbb{P}_p(F_n) \xrightarrow{n \rightarrow \infty} 1$. $\exists N$ $\mathbb{P}_p(F_N) > 0$

$\mathcal{Q} = \infty$



Lemma For any configuration with N particles, the number of four points in $[-N, N]^d$ is at most $|\partial[-N, N]^d|$.

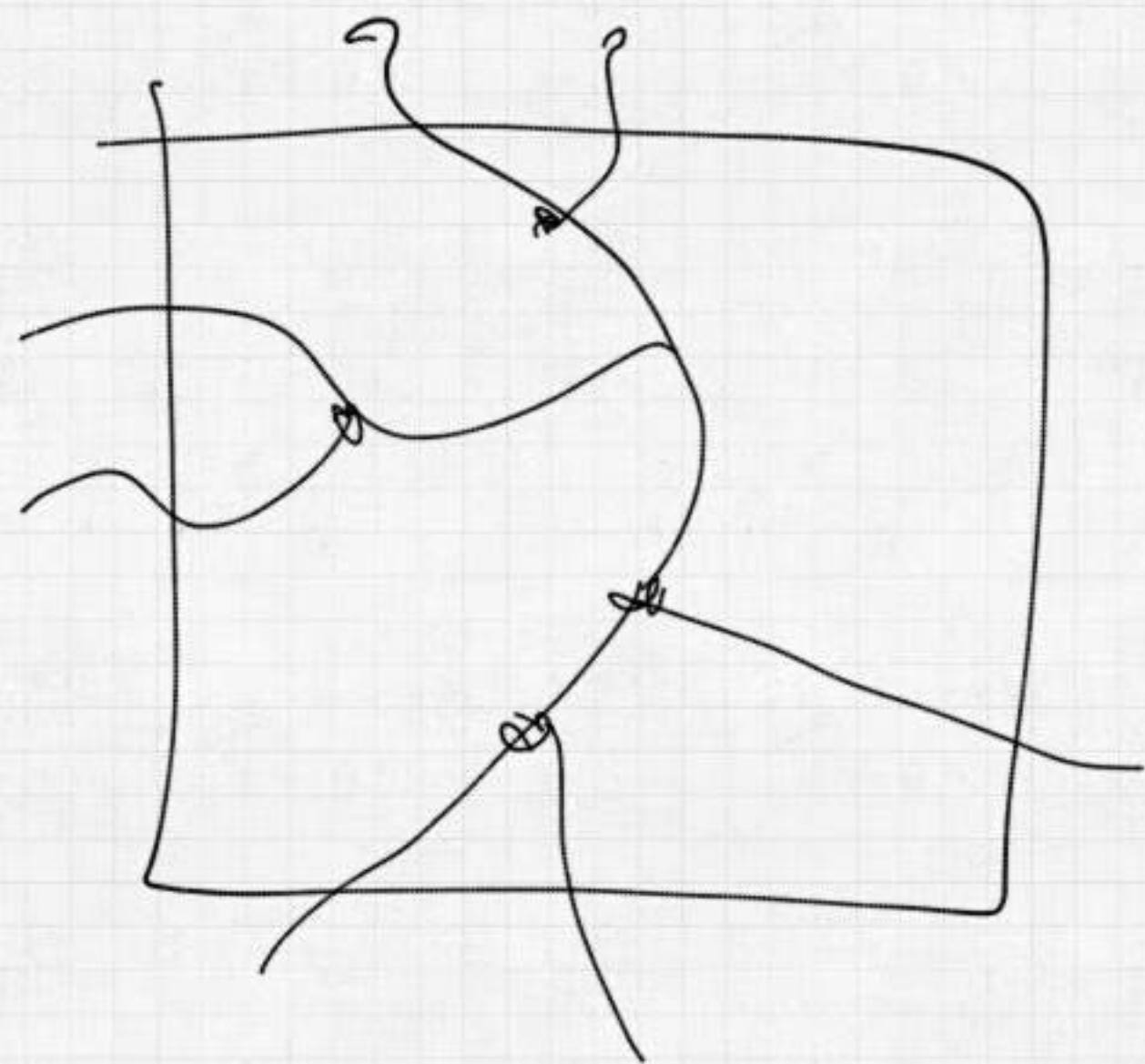
Finishing the proof given the Lemma: Let $f = \rho(0)$ is a four point

choose N large s.t. $f(2N+1)^d \gg |\partial[-N, N]^d|$.

by Lemma # four points $\leq |\partial[-N, N]^d|$

but $\mathbb{E}[\# \text{ four points}] = \mathbb{E}\left[\sum_{x \in [-N, N]^d} \mathbb{1}_{x \text{ is a four point}}\right] = f(2N+1)^d$ $\left\{ \begin{array}{l} \text{ } \\ \text{ } \\ \text{ } \end{array} \right.$
 \square

Proof of Lemma 1



1. \forall finite set S of graph points in the same int cluster of one set removing it keeps all the rest in the same int cluster.

2. (induction using 1). \forall finite set S of graph points in the same int cluster if we remove all elements of S we break the int cluster to at least $|S| + 2$ clusters.

Continuity of percolation function:

Ex: $\theta_d(p)$ is a right continuous function of p on $[0,1]$ (Thm 2.5 Steif).
steif

Thm: $\lim_{\pi \uparrow \tilde{p}} \theta(\pi) = \theta(\tilde{p})$.

Proof: Couple all percolation realizations on the same space.
 $\chi(e) = \{e \in E^d\}$ and $\mathcal{N}(u)$. an edge is p -open if $\chi(e) < p$.

$$\uparrow \text{ If } p_1 < p_2 \quad \{e: e \text{ is } p_1 \text{ open}\} \subseteq \{e: e \text{ is } p_2 \text{ open}\}.$$

Let C_p - p -cluster of origin $\Rightarrow C_{p_1} \subseteq C_{p_2}$ (increasing)

$$\lim_{\pi \uparrow \tilde{p}} \theta(\pi) = \lim_{\pi \uparrow \tilde{p}} \mathbb{P}(|C_\pi| = \infty) = \mathbb{P}\left(\bigcup_{\pi < \tilde{p}} |C_\pi| = \infty\right) = \mathbb{P}(|C_{\tilde{p}}| = \infty \text{ for some } \pi < \tilde{p})$$

Since $\{ \pi \in \tilde{\rho} \mid |C_\pi| = \infty \} \subseteq \{ \pi \mid |C_\pi| > \infty \}$ we need
 to show $\mathbb{P}(\{ \pi \mid |C_\pi| = \infty \} \cap \{ \pi \mid |C_\pi| < \infty \vee \pi \in \tilde{\rho} \}) = 0$.

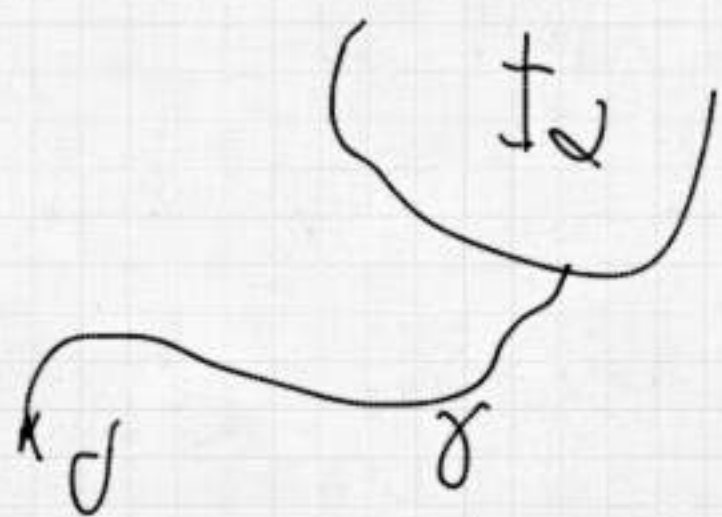
Let $\alpha \in \rho_c < \alpha < \tilde{\rho}$ then by uniqueness the set α -cluster

$\mathbb{I}_\alpha \subseteq C_\rho$ a.s. If $0 \in \mathbb{I}_\alpha$ choose $\pi = \alpha$. If not

there is a path $\gamma: 0 \rightarrow \mathbb{I}_\alpha$ $\tilde{\rho}$ -open. Let

$$\mu := \max\{X(e) : e \in \gamma\}$$

$$\mu < \tilde{\rho}.$$



$$\{ \text{LCC in } \mu < \pi < \tilde{\rho} \Rightarrow |C_\pi| = \infty$$



FKG-inequality: For $(V, \mathcal{A} = \{A_i\}_{i \in J})$, there is
a partial order $w \leq w'$ if $w_i \leq w'_i \forall i \in J$.

Definition: $f: \{0,1\}^J \rightarrow \mathbb{R}$ is increasing if $w \leq w' \Rightarrow f(w) \leq f(w')$.

Example: open path between two vertices is increasing.

Theorem Let f, g be increasing functions, then $(f, g \in L^2)$

$$\mathbb{E}[fg] \geq \mathbb{E}[f]\mathbb{E}[g].$$

Example s.t. $x, y \in V(G)$, $A = \{\exists \text{ open path } x \rightarrow y\}$
 $B = \{\checkmark z\}$

$$P(A \cap B) \geq P(A)P(B).$$

Proof. Assume J finite (approximate otherwise).

Induction on $|J|$. If $|J|=1$, since f, g increasing

$$(f(w_1) - f(w_2))(g(w_1) - g(w_2)) \geq 0.$$

Using expectation

$$\mathbb{E}[f(w_1)g(w_1)] + \mathbb{E}[f(w_2)g(w_2)] - \mathbb{E}[f(w_2)g(w_1)] - \mathbb{E}[f(w_1)g(w_2)] \geq 0$$

$$\Rightarrow 2\mathbb{E}[fg] \geq 2\mathbb{E}[f]\mathbb{E}[g].$$

Assume true for $|J|=k-1$ on f, g functions of k variables.

$$\mathbb{E} [f(x_1, \dots, x_n) g(x_1, \dots, x_n)] = \mathbb{E} [\mathbb{E} [\dots \mid x_1, \dots, x_{n-1}]]$$

$n=1$ case

$$\mathbb{E} \left[\mathbb{E} [f(x_1, \dots, x_n) \mid x_1, \dots, x_{n-1}] \mathbb{E} [g(\cdot) \mid x_1, \dots, x_{n-1}] \right]$$

induction hypothesis

$$\mathbb{E} [f] \mathbb{E} [g] \quad \square$$

BH inequality: we would like to have a negative correlation inequality, $A \cap B$ is replaced with A and B occur on disjoint edge sets: $A \cap B = \emptyset$.

let $\{(\Omega_i, \mathcal{F}_i, \mu_i)\}_{i \in V}$ be finite probability spaces

and $(\Omega, \mathcal{F}, \mu)$ be the product.

Two configurations $w = (w_1, \dots, w_n) \in \Omega$ are said to be equal on $S \subset [n]$

if $w_i = \tilde{w}_i \forall i \in S$.

Definition: Let $S \subset [n]$, and let $S^c = [n] \setminus S$. An event $A \in \mathcal{F}$ is said to occur on the set S in the configuration w if A occurs using only the r.v. over S .

$$A|_S = \{w: \exists \tilde{w} \supseteq w \text{ on } S \Rightarrow \tilde{w} \in A\}.$$

Two events $A_1, A_2 \in \mathcal{F}$ are said to occur disjointly, denoted $A_1 \circ A_2$ if there are two disjoint sets on which they occur.

$$A_1 \circ A_2 = \{w: \exists S_1, S_2 \subset [n], S_1 \cap S_2 = \emptyset, \text{ with } A_1|_{S_1} \cap A_2|_{S_2}\}.$$

BK - Reiner: $\mu(A \cup B) \leq \mu(A) \mu(B)$.

Example: 1. Let $G \subset \mathbb{Z}^d$ finite

$A_G(x, y) = \{ \exists \text{ open path connecting } x \rightarrow y \}$.

the even $A_G(x, y) \cap A_G(u, v) = \{ \exists \text{ two edge disjoint open paths } \}$
connecting $x \rightarrow y$ and $u \rightarrow v$.

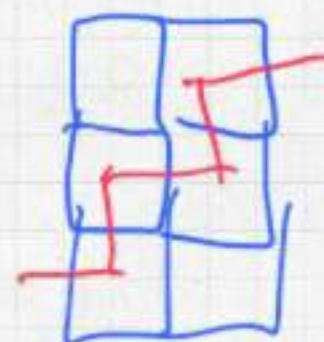
$$\mu(A_G(x, y) \cap A_G(u, v)) \leq \mu(A_G(x, y)) \mu(A_G(u, v)).$$

Distances in percolation:

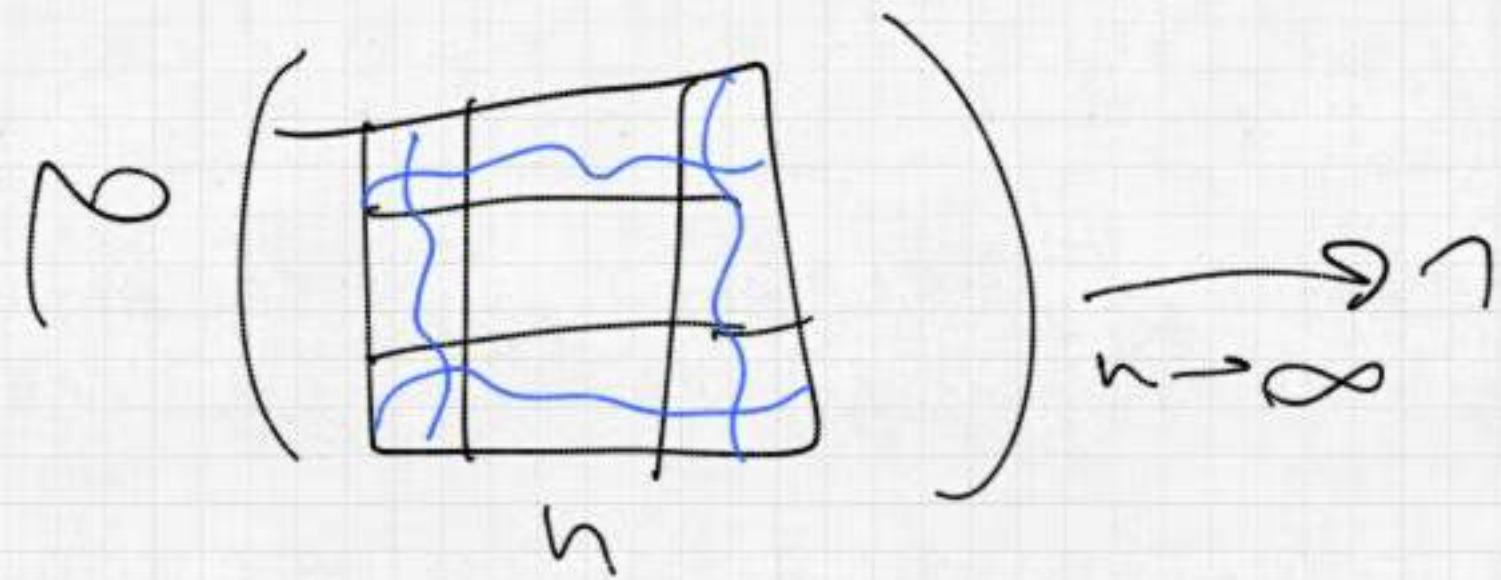
Lemma $P \approx P_c(\mathbb{Z}^2)$, $P\left(\text{rectangle with } h \text{ height and } ah \text{ width}\right) \xrightarrow{h \rightarrow \infty} 1$

Proposition $P(\text{rectangle with wavy path}) \stackrel{FHG}{\geq} P(\text{rectangle with straight path})^2$

$P(\text{rectangle } C) = P(\text{rectangle with dual closed path longer than } n) \leq e^{-cn}$



BS FKG



Domination by product measure: (Liggett, Schonmann, Stacey)

Let $(X_s)_{s \in \mathbb{Z}^d}$ be a collection of pairs r.v. Let say

that (X_s) is k -dependent if $\{X_s : s \in A\}, \{X_s : s \in B\}$
are independent whenever $\text{dist}(A, B) > k$.

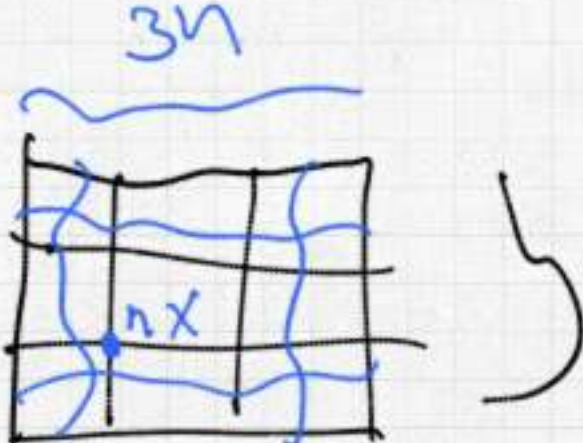
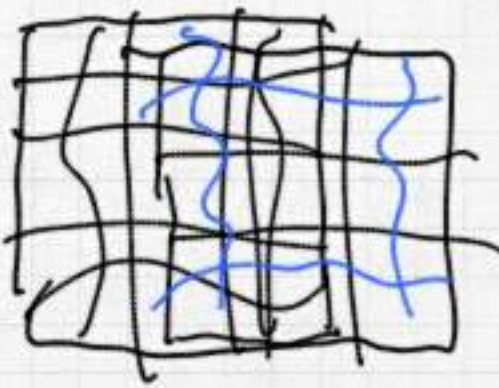
Lemma [LSS 97] Let $d, k \geq 1$. \exists non decreasing function $\pi: [0,1] \rightarrow [0,1]$

Satisfying $\pi(d) \rightarrow 1$ as $d \rightarrow 1$ s.t. if (X_s) is k -dependent and $P(X_s=1) \geq d \forall s \in \mathbb{Z}^d$ then (X_s) is dominated (stochastic) from below by iid percolation with parameter $\pi(d)$.

Definition: Let $Y = \{Y_s : s \in \mathbb{N}\}$, $Z = \{Z_s : s \in \mathbb{N}\}$ be families of iid variables.

We say Y stochastically dominates Z , $Y \succeq_{st} Z$, if

$$\mathbb{E}[f(Y)] \geq \mathbb{E}[f(Z)] \quad \& \text{ bounded increasing } f: \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{R}.$$

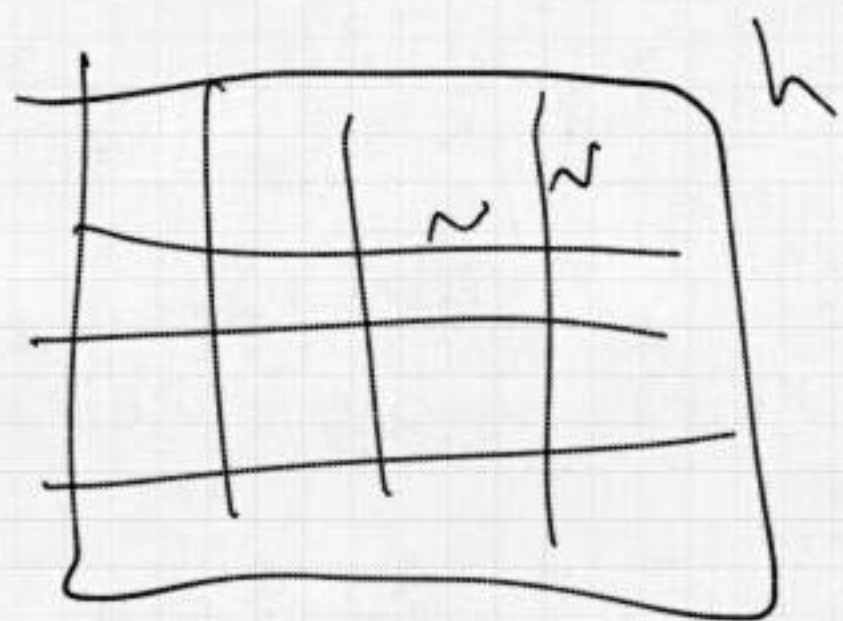
Let $X_x = \mathbb{1}_{\left\{ \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \right\}}$  


Th 1 $\forall \rho > \rho_c(\mathbb{Z}^d) \exists c, c' > 0$ s.t. $\forall h$ large then

$$P_p(|C_d(h)| > c, h^d) < e^{-c'h}$$

Proof $\rho > \rho_c$ \Rightarrow $\rho_c(\mathbb{Z}^d) < \rho < 1$ s.t. $\forall \rho > \rho_c$

$$P_p(|C_d(h)| > \tilde{c} h^d) < e^{-c'h}$$



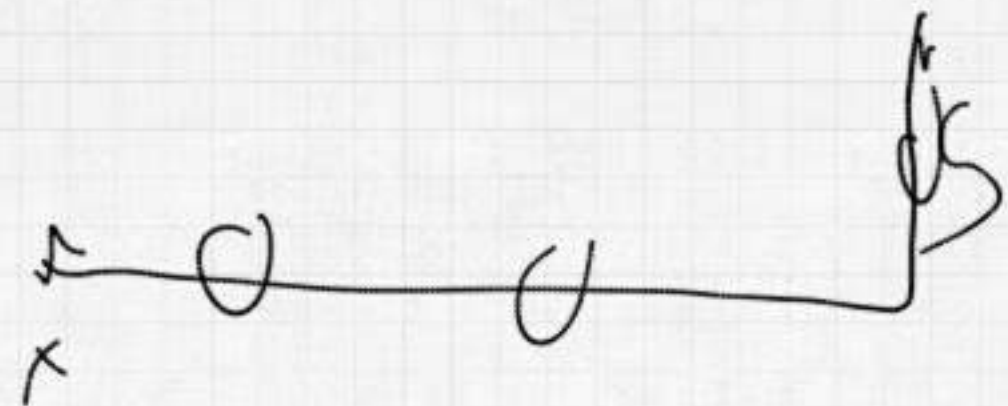
$h, h \uparrow$
 $C_1(h) > \tilde{c} \left(\frac{h}{n}\right)^d$
 \Rightarrow $h \rightarrow \infty$ 

Lemma If $1-p < p_c$ then $d_w(x, S) < \infty$

$k \times k$

\llcorner
positive
norm

Proof



Let A be size of closed components

$$|E(A)| \leq \sum_{i=1}^k |A_i| = C$$

\uparrow
where size.

$$N(d(x, S) \geq k) \leq \frac{|E(A)|}{k} = \frac{C}{k}$$

$\xrightarrow{\text{of } B(h)}$ Left right crossing: $B(h) = [-h, h]^d$
 crossing is on open path of $B(h)$ joining same vertex x
 with $x_i > -h$ to same y with $y_i = h$. Define $LR(h)$ to
 event of a crossing.

Theorem If $\theta(p) > 1$ then $\mathbb{P}_p(LR(h)) \rightarrow 1$ as $h \rightarrow \infty$.

Proof (using uniqueness) Let $I(h) = \{C \cap B(h) \neq \emptyset\}$
 let $\varepsilon > 0$, choose h large st

$$\mathbb{P}_p(I(h)) > 1 - \varepsilon \quad (\text{why can we? ergodic theory}).$$

For $h > h_0$ let f_1, \dots, f_{2d} be the faces of $B(h)$. If
 $I(h)$ occurs then some vertex in $B(h)$ is connected to
 a face.

$$1 - \mathbb{P}_p(\mathbb{I}(m)) \geq 1 - \mathbb{P}_p\left(\bigcup_{i=1}^m \{B(m) \leftrightarrow F_i \text{ in } B(m)\}\right)$$

$$= \mathbb{P}_p\left(\bigcap_{i=1}^m \{B(m) \leftrightarrow F_i \text{ in } B(m)\}^c\right)$$

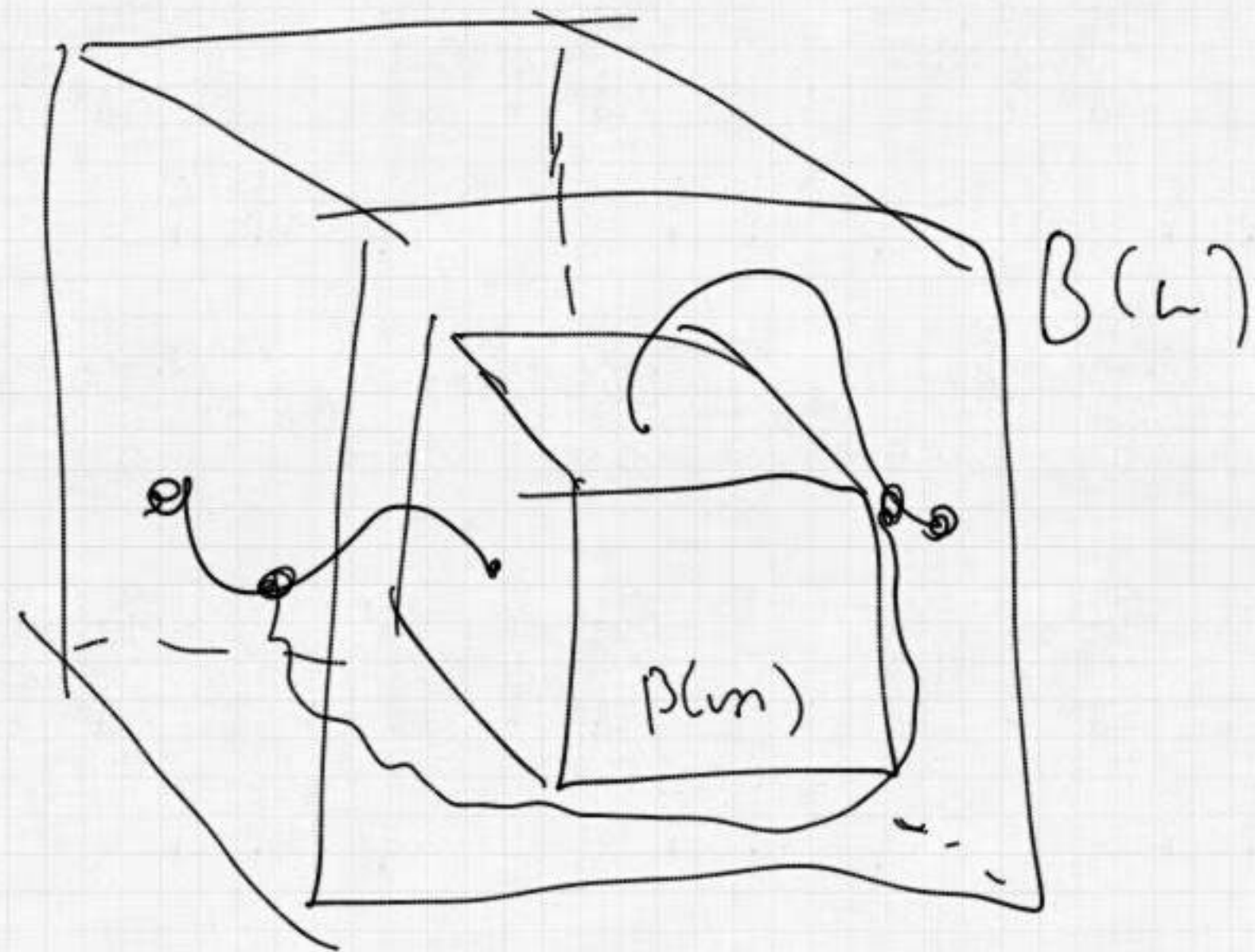
$\stackrel{FKK}{\geq} \left(1 - \mathbb{P}_p(B(m) \leftrightarrow F \text{ in } B(m))\right)^m$

same chosen face.

$$\Rightarrow \mathbb{P}_p(B(m) \leftrightarrow F \text{ in } B(m)) \leq 1 - \epsilon^{1/m}$$

$$\Rightarrow \mathbb{P}_p(\{B(m) \leftrightarrow F_L \text{ in } B(m)\} \cap \{B(m) \leftrightarrow F_R \text{ in } B(m)\}) \leq (1 - \epsilon^{1/m})^2$$

last face



Let $A_{min} = \{ \exists \text{ two vertices of } \partial B(m) \text{ in disjoint open clusters of } B(m) \text{ both intersect } \partial B(h) \}$,

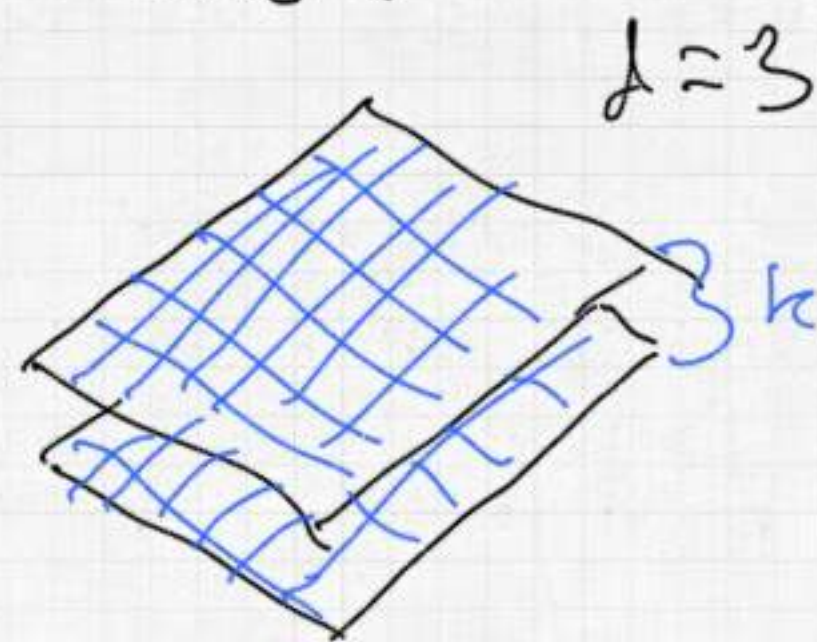
Since $A_{min} \supseteq A_{m,h+1}$ $\rho_p(A_{min}) \xrightarrow{h \rightarrow \infty} \rho_p(\exists \text{ two vertices of } \partial B(m) \text{ is disjoint via clusters}) = 0$

$\Rightarrow \rho_p(LR(h)) \geq (1 - \varepsilon^{h+2}) \rho_p(A_{min}) \geq 0$ \square

Percolation in Slabs:

For $d \geq 3$ define the $2d$ -slab of thickness k

$$S_k = \mathbb{Z}^2 \times \{0, 1, \dots, k\}^{d-2}$$



Since $S_k \subseteq S_{k+1} \subseteq \dots \subseteq \mathbb{Z}^d$

$p_c(S_k) \geq p_c(S_{k+1}) \geq p_c$. Bounded decreasing thus converges

$$p_c^{\text{slab}} = \lim_{k \rightarrow \infty} p_c(S_k)$$

Theorem: Grimmett, Marstrand 1990!

$$\hookrightarrow d \geq 3 \quad p_c^{\text{slab}} = p_c$$

Let $S_n(L) = [-h, h]^2 \times [0, L]^{d-2}$
 $T_n(L) = [-h, h]^{d-1} \times [0, L]$.

Lemma Let $d \geq 3$ and $\rho > \rho_c$. There exists a positive integer L and a $d(\rho, L) > 0$ s.t

- (1) $\rho_p(x \leftrightarrow y \text{ in } S_n(L)) \geq d$ $\forall x, y \in S_n(L)$ and $h \geq 1$.
- (2) $\rho_p(x \leftrightarrow y \text{ in } T_n(L)) \geq d$ $\forall x, y \in T_n(L)$ and $h \geq 1$ (HL Grimmett \rightarrow AF partial args).

Proof: Since $\rho > \rho_c$, by Grommett Marstrand $\exists L$ s.t.

$$\rho > \rho_c \left(\mathbb{Z}_+^2 \times [0, L]^{d-2} \right).$$

Let $\rho > \rho' > \rho_c \left(\mathbb{Z}^2 \times [0, L]^{d-2} \right)$ Define $\theta = \rho_{\rho'} \left(0 \leftrightarrow \infty \text{ in } \mathbb{Z}_+^2 \times [0, L]^{d-2} \right) > 0$

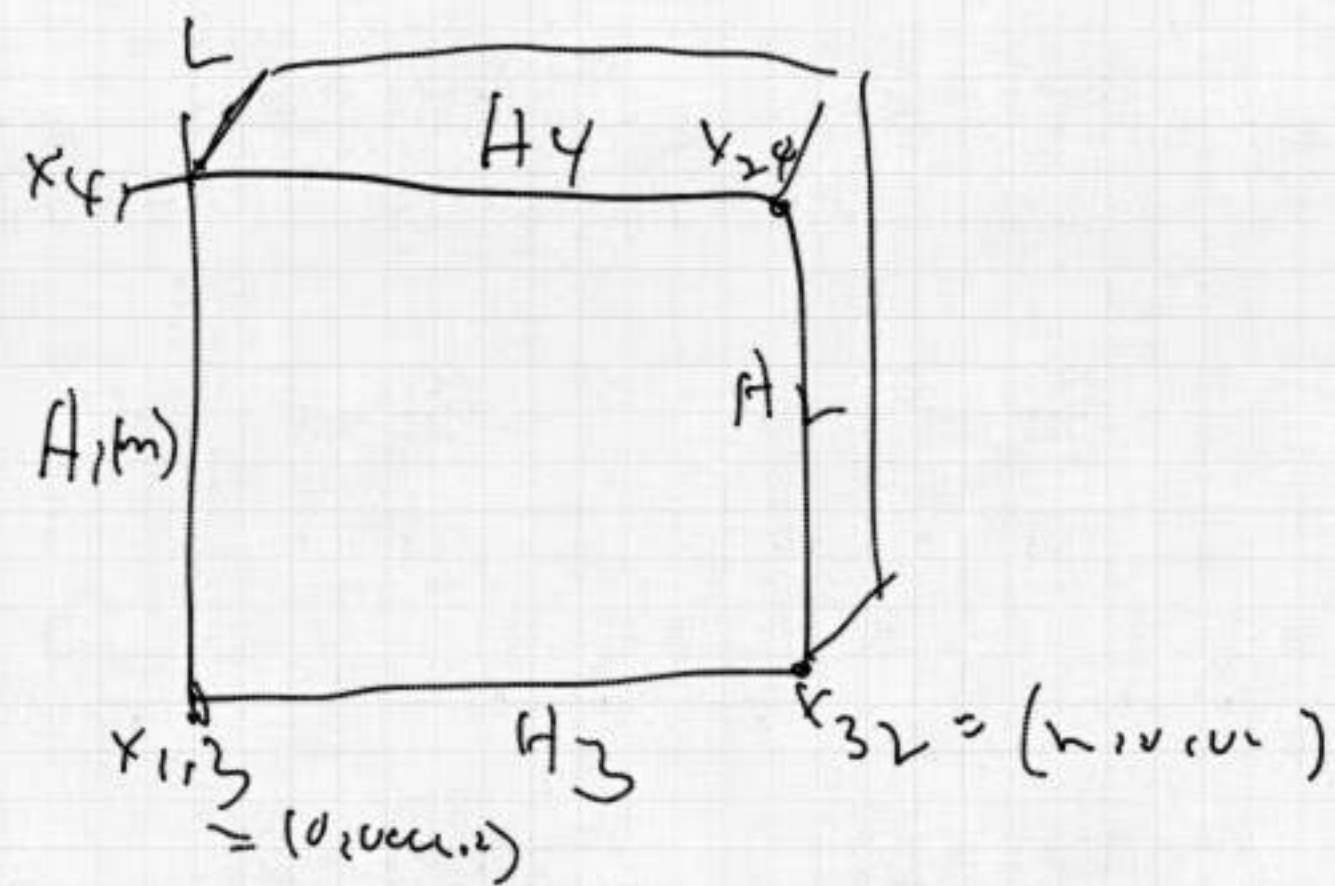
Define on $U_m(L) = [0, L]^2 \times [0, L]^{d-2}$

$$\theta = \rho_{\rho'} \left(0 \leftrightarrow H_2(h) \cup H_4(h) \text{ in } U_m(L) \right)$$

$$\leq \rho_{\rho'} \left[0 \leftrightarrow H_2(h) \text{ in } U_m(L) \right]$$

$$+ \left[0 \leftrightarrow H_4(h) \right]$$

$$\Rightarrow \rho_{\rho'} \left(0 \leftrightarrow H_2(h) \text{ in } U_m(L) \right) \geq \frac{1}{2} \theta$$



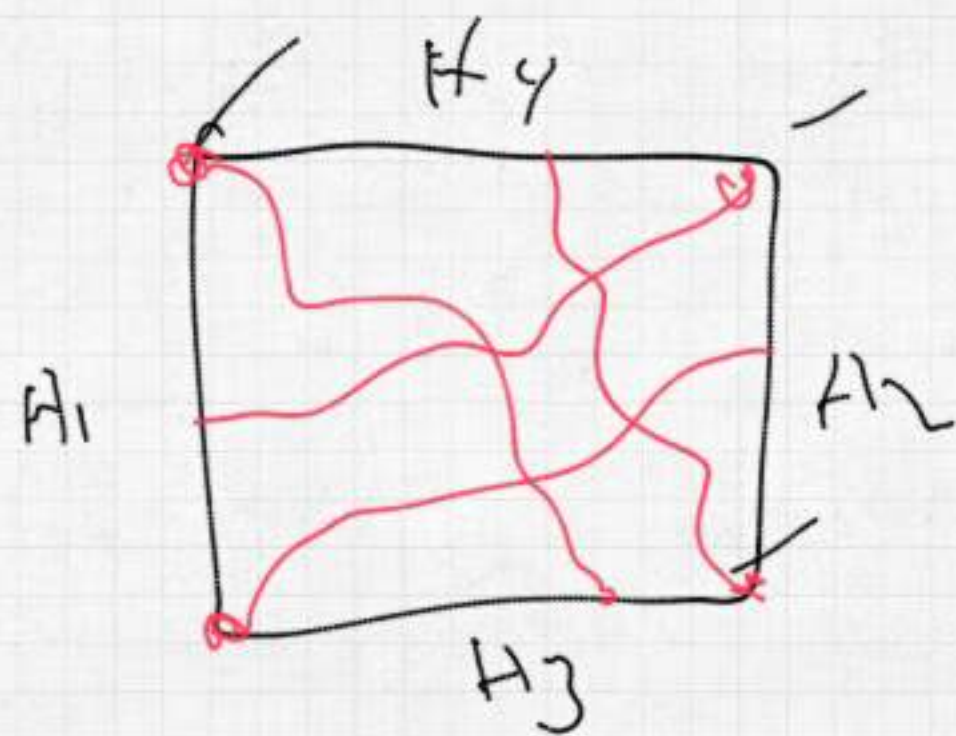
By rotation invariance the probability of all the following events $\geq \frac{1}{2} \theta$

$$A_{13} = \{X_{13} \leftrightarrow H_2 \text{ in } U_m(L)\}$$

$$A_{32} = \{X_{32} \leftrightarrow H_4 \text{ in } U_m(L)\}$$

FKG

$$P_p(A_{13} \cap A_{32} \cap A_{24} \cap A_{41}) \geq \left(\frac{1}{2}\theta\right)^4$$



there is a set of at least $4(d-2)L$ distinct open $X_{13} \leftrightarrow$

$$X_{32} \leftrightarrow X_{24} \leftrightarrow X_{41} \text{ in } U_m(L) \Rightarrow \exists \delta_i(A, L) > 0 \text{ s.t.}$$

$$P_p(X_{13} \leftrightarrow X_{32} \text{ in } U_m(L)) \geq \delta_i \quad \forall m \geq 1.$$

now by FKG it is enough to show

$$P_p(0 \leftrightarrow z \text{ in } S_L(L)) \geq c_2 > 0 \quad \forall z \in S_{in}(L).$$

Let $z = (z_1, z_2, \dots, z_d) \in S_{in}(L)$. assume ^{wlog} $0 \leq z_1 \leq z_2$

Define the boxes,

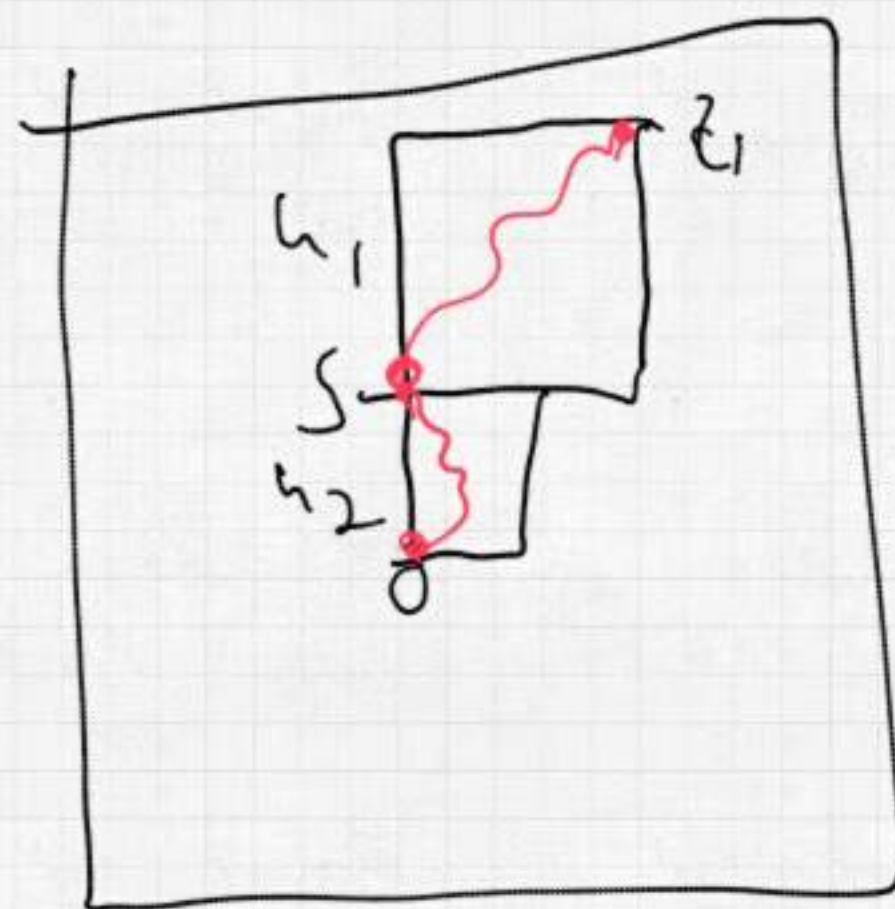
$$U_1 = [0, z_1] \times [z_2 - z_1, z_2] \times [0, L]^{d-2}$$

$$U_2 = [0, z_2 - z_1] \times [0, L]^{d-2}$$

by FKG

$$P_p(0 \leftrightarrow z \text{ in } S_{in}(L)) \geq P_p(z \rightarrow S \text{ in } U_1) P_p(S \leftrightarrow 0 \text{ in } U_2)$$

$(0, z_2 - z_1, 0, \dots)$



also

$$P_p(z \leftrightarrow y \text{ in } \mathcal{H}_1) \geq p^{(d-2)L} P_p(z' \leftrightarrow y \text{ in } \mathcal{H}_1) \text{ where}$$

$z' = (z_1, z_2, \dots)$ (order of length $(d-2)L$ connecting them,

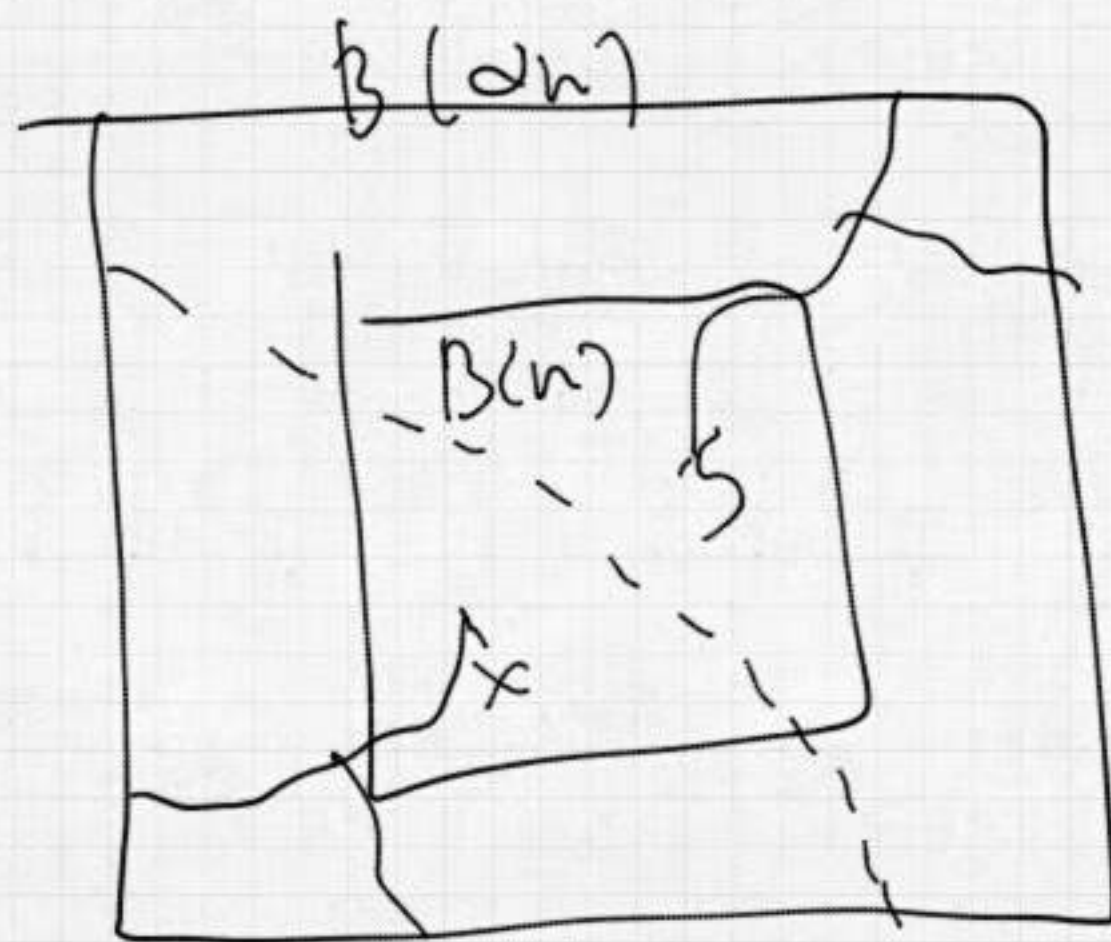
$$\Rightarrow P_p(z \leftrightarrow 0 \text{ in } S_n(L)) \geq p^{(d-2)L} \delta_1 z$$

Lemma 2.7

Lemma 1, (HW) Let $d \geq 3$, $r \geq \rho$, $\partial \Omega = \{r\} \times \mathbb{S}^{d-1}$ s.t

$$\prod_{\rho} \left(x \in \partial B(\alpha h), y \in \partial B(\alpha h), x \neq y \text{ in } \partial B(\alpha h) \right) \leq e^{-h(\alpha-1)}$$

$\forall x, y \in \partial B(h), h \geq 1, \alpha \geq 1$



Chemical Distance:

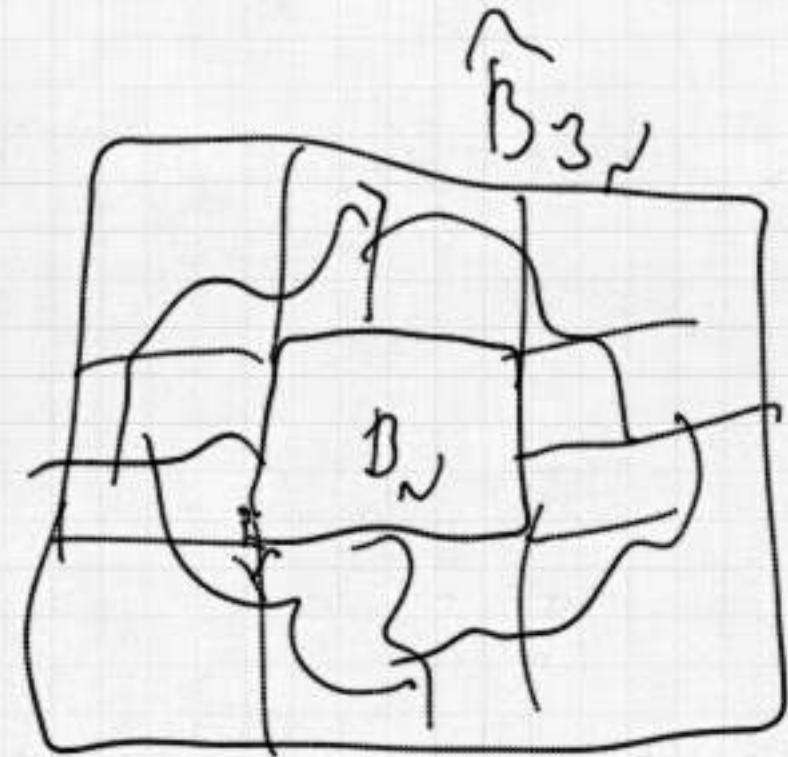
Denote $B_N(x) = x + \mathbb{Z}^d \cap [0, N]^d$

and $\tilde{B}_{3N}(x) = x + \mathbb{Z}^d \cap [-N, 2N]^d$

Let $G_N(x)$ be the event:

① $\forall x$ neighbor of y the side of the block $B_N(Ny)$ adjacent to $B_N(Nx)$ is connected to the opposite side of $B_N(Ny)$ by an open path.

② every two open paths connecting $B_N(Nx)$ to $\partial \tilde{B}_{3N}(Nx)$ are connected in $\tilde{B}_{3N}(Nx)$.



We saw $\mathbb{P}_p(G_N(u)) \xrightarrow[\infty]{} 1$ (was $\mathbb{P}_p(G_N(x)) = \chi(x, \mathbb{Z}^d)$)
dominated from below by i.i.d percolation with
parameter $\eta_N(p)$,

for two points x, y we want to find

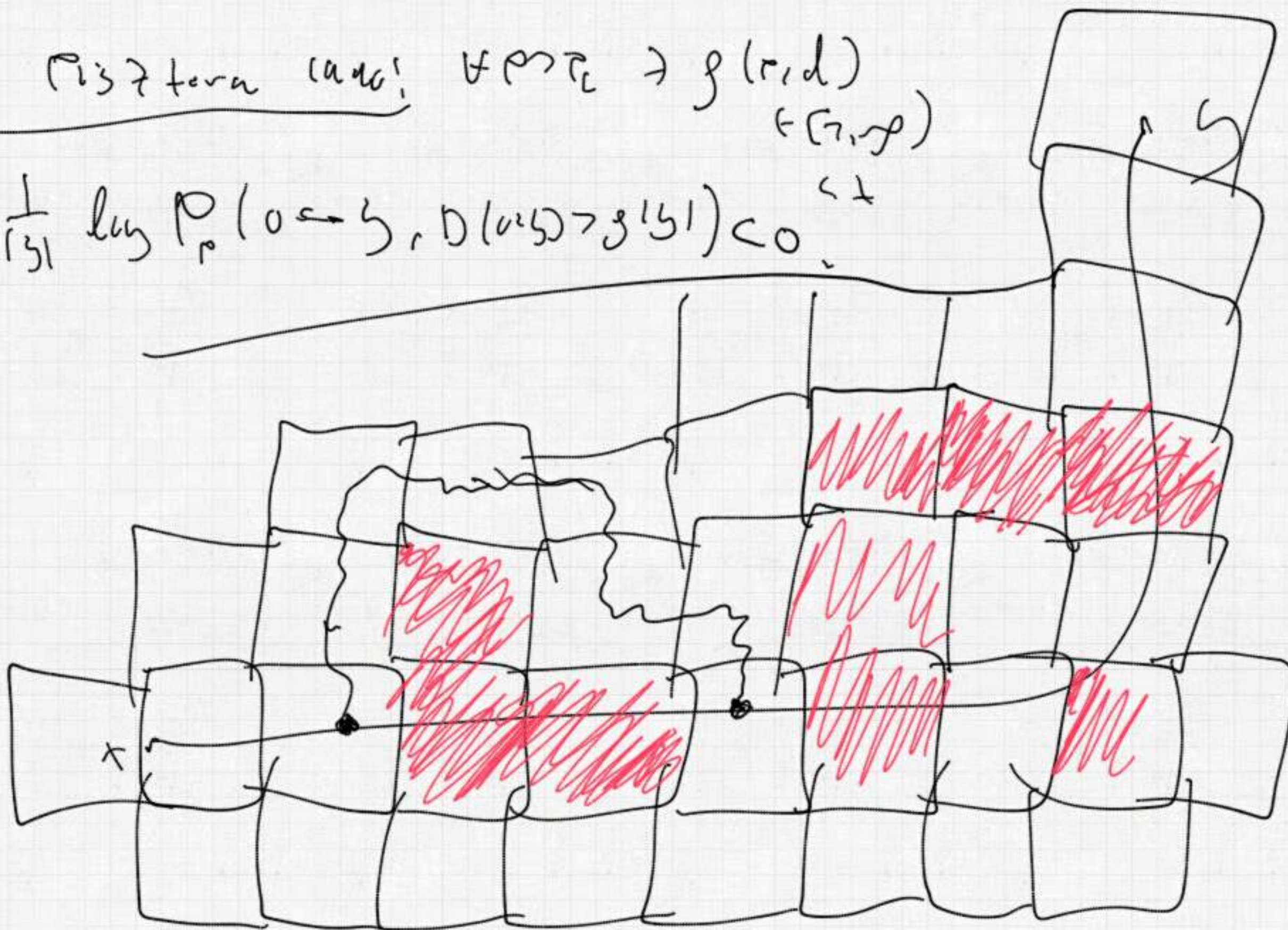
an open path connecting them. If $\|x-y\|_1 = h$

we have at least h/N boxes of size N intersecting

an L_1 -path connecting them.

$\forall \epsilon > 0$ there exists n_0 such that $n > n_0 \Rightarrow \frac{1}{n} \log P_p(0 \rightarrow \gamma, D(\omega_\gamma) > \epsilon |\gamma|) < \epsilon$

$\lim_{|\gamma| \rightarrow \infty} \frac{1}{|\gamma|} \log P_p(0 \rightarrow \gamma, D(\omega_\gamma) > \epsilon |\gamma|) < \epsilon$



Let $\{C_i\}$ be the collection of ball connected macro-clusters
intersecting γ ,

$$|\gamma'| \leq (3r)^d \left(|\gamma| + \sum_i |C_i| \right)$$

$$\sum_i |C_i| \leq \sum_i |C_i^{\text{in}}|$$

Then Large Deviations,

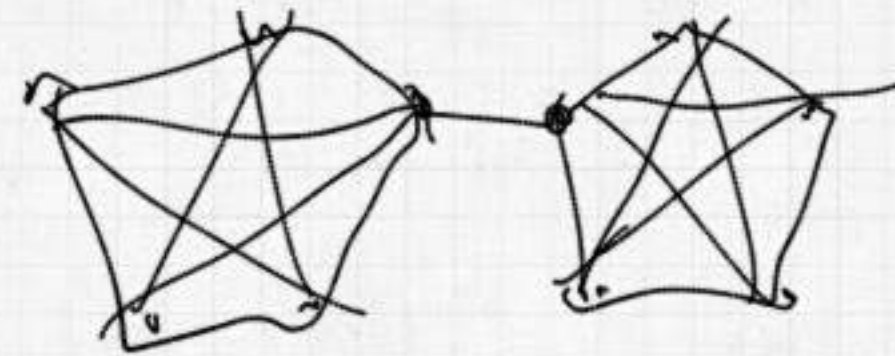
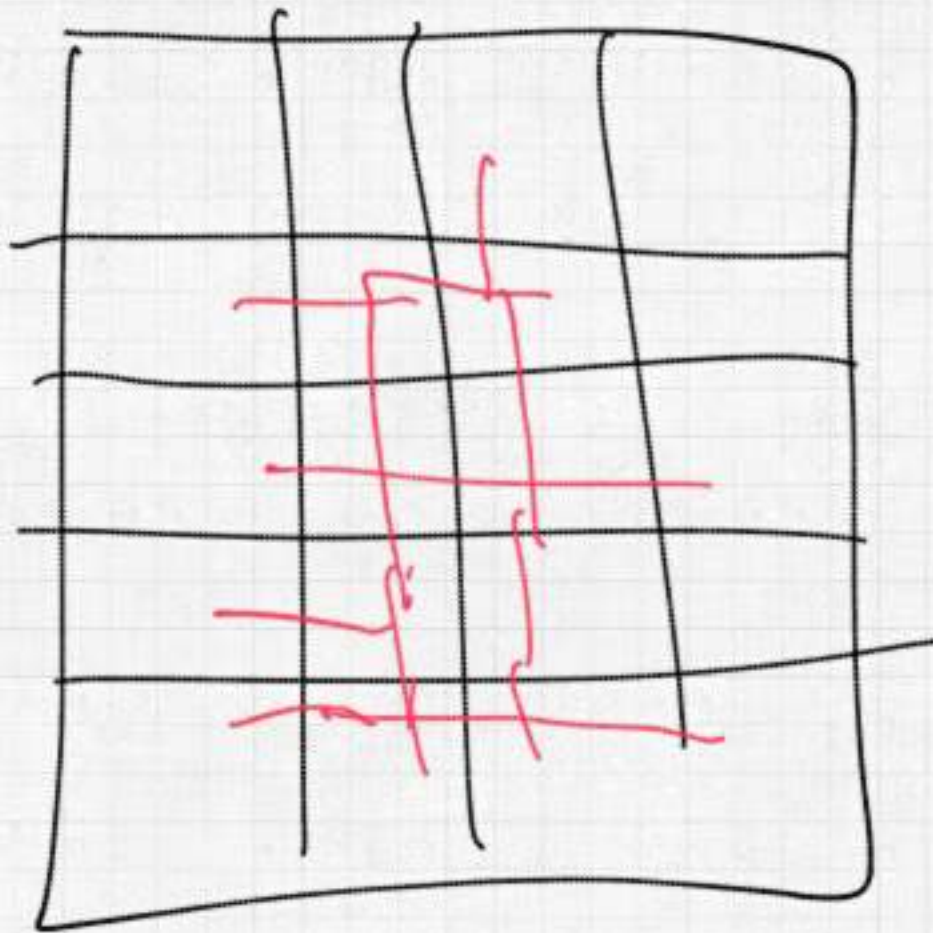
\downarrow
 $\frac{1}{\epsilon} \rightarrow 0$
Here $\rightarrow 0$.

Isoperimetry:

for a finite graph

G

$$\mathcal{L}_G = \text{ring} \left\{ \frac{|DA|}{|A|} : |A| < \frac{|G|}{2} \right\}$$



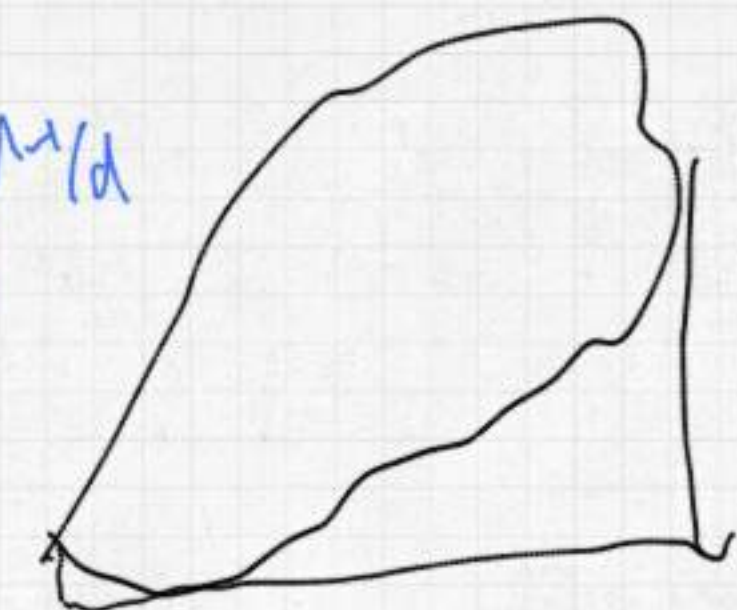
Isoperimetry of \mathbb{R}^2 .



$(2a + 2b) \geq 2\sqrt{4ab}$

$|\partial A| \geq |\partial \square(A)|$

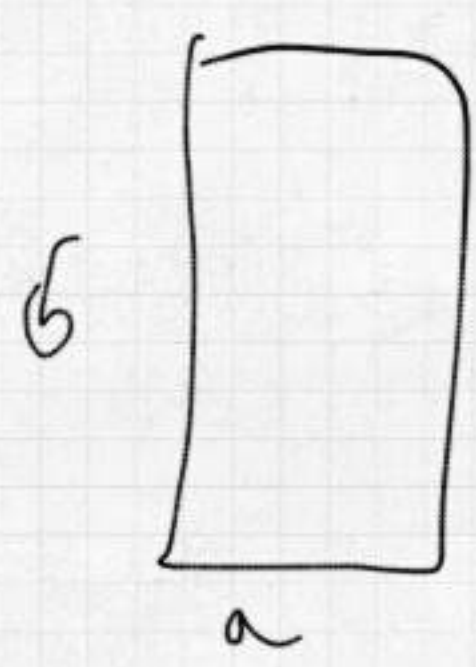
$\geq |\partial A|^{M/A} \geq |A|^{M/A}$



larger value smaller boundary. Contract to have the same volume the boundary will be smaller still.

$(x^h)' = h x^{h-1}$

we get a rectangle with smaller cheeger.



$ab = h^2$
 $|\partial A| = 2a + 2b \geq 4h$

$a \geq \frac{h^2}{b}$

$b > a$
 $(b > h)$

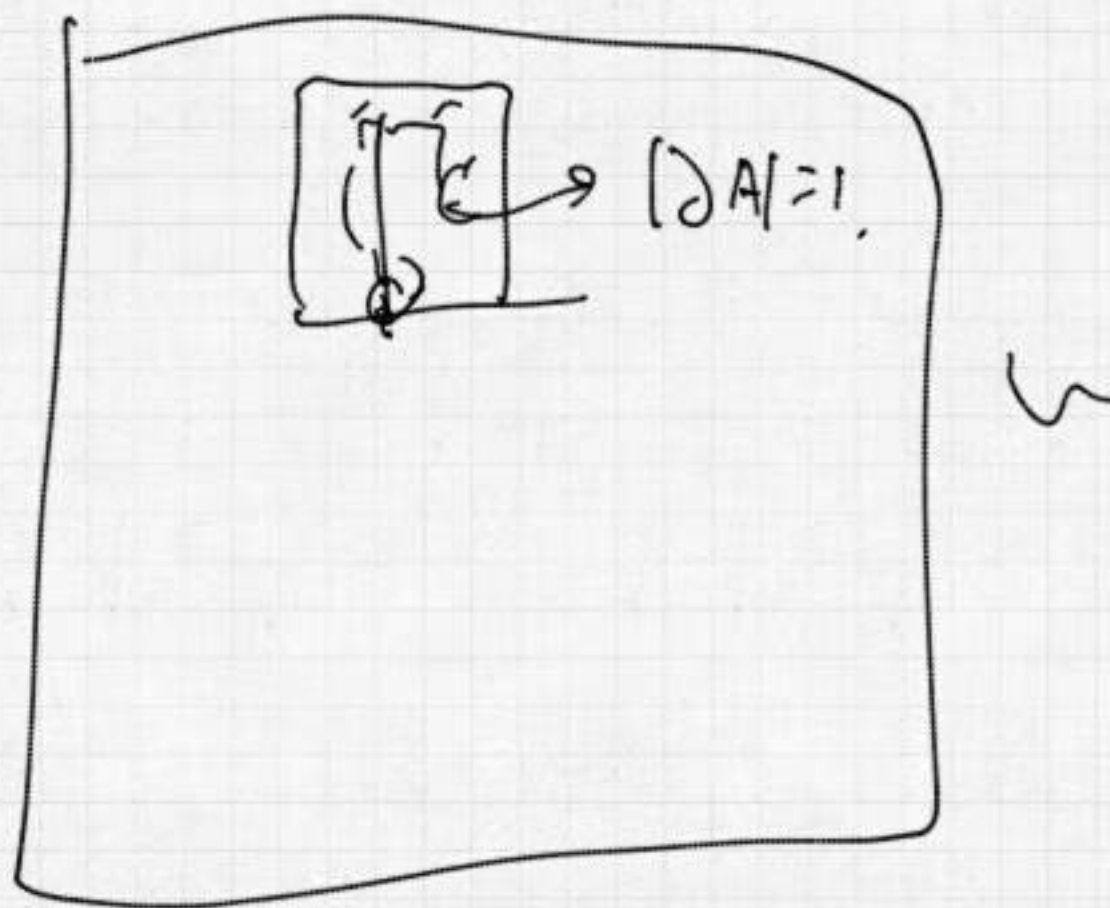
$2a + 2b = 2\frac{h^2}{b} + 2b$

$()' = -2h^2 b^{-2} + 2 = 0$

$b = h$

What about Percolation?

If we look at a box
we need to restrict ourselves
to "big" sets.



BIBL 08

Lemma $\forall d \geq 2, \rho > \rho_c(d) \exists c_1(d), c_2(d)$ and an a.s.
finite r.v. R s.t. $\forall R > R_0$ and each w -connected $A \subset \mathbb{Z}^d$
 $A \subset C_\infty \cap [-R, R]^d$ and $|A| \geq (c_1 \log R)^{d/(d-1)}$

we have

$$|\partial^v A| \geq c_2 |A|^{d-1/d}$$

Given a finite set

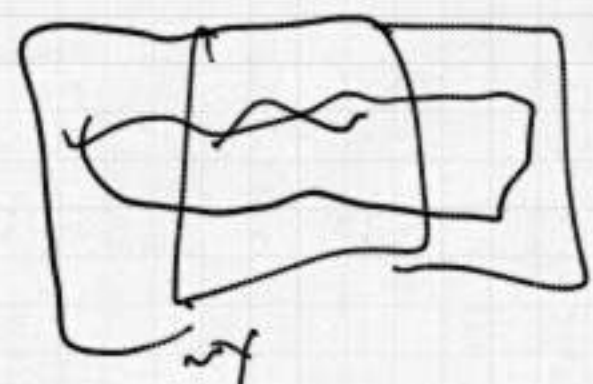
$\Delta \subset \mathbb{Z}^d$, let $\Delta^{(N)} = \{x \in \mathbb{Z}^d : \Delta \cap B_N(x) \neq \emptyset\}$
 $\overline{\Delta}^{(N)}$ complement of the inf 'comp of $\Delta^{(N)}$,
 inner site boundary $= \partial^* \Delta = \{x \in \Delta : \exists y \in \mathbb{Z}^d \setminus \Delta \text{ with } |x-y|=1\}$,

Lemma: For $v \in \mathcal{N}$, let $\Delta \subset \mathbb{Z}^d$ be v -connected with
 $\overline{\Delta}^N = \Delta$ and $\text{diam}_{\infty} \Delta \geq 3N$. If $|\partial^v \Delta| < \frac{1}{2 \cdot 3^d} |\partial^* \Delta|$ then

$$|\{x \in \partial^v \Delta : \text{C}_v(x) \text{ occurs}\}| > \frac{1}{2} |\partial^* \Delta|.$$

Proof:

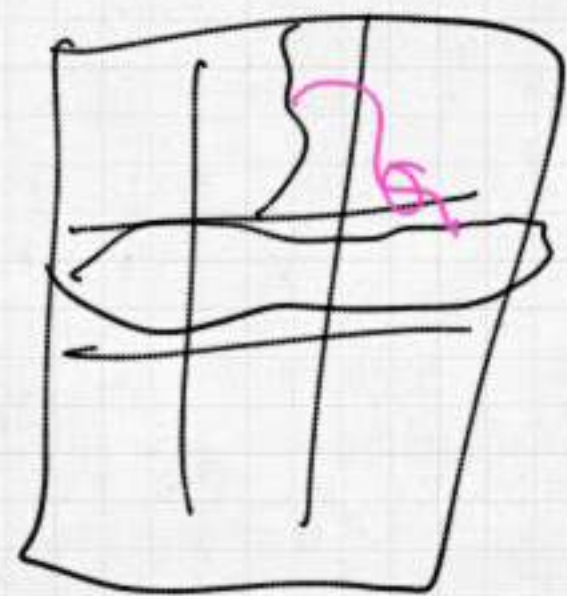
$\forall x \in \partial^x \Delta$ then $\Delta \cap B_N(x) \neq \emptyset$. We claim $\forall x \in \partial^x \Delta$
 $G_N(x) \subset \tilde{B}_{3N}(x)$ contains an edge in $\partial^w \Delta$



If $G_N(x)$ occurs, by diam $\Delta > 3N$ $B_N(x)$

is connected to $\partial \tilde{B}_{3N}(x)$ by an open path in Δ .


by definition of $G_N(x) \Rightarrow$ another crossing to a $B_N(y) \cap \Delta = \emptyset$.



these paths are connected by
 a path in $\tilde{B}_{3N}(x)$ which must
 contain a boundary edge,

Since each edge in $\partial^w \Delta$ belongs to at most $3^d \hat{B}_{3^{-d}}(x)$ cubes, with $x \in \partial^* \Delta$ the # of $\partial^* \Delta$ where $G_{\nu}(x)$ occurs is bounded by $3^d |\partial^w \Delta|$,

$$\Rightarrow |\partial^* \Delta| - |\{x \in \partial^* \Delta, G_{\nu}(x) \text{ occurs}\}| \leq 3^d |\partial^w \Delta| \stackrel{\text{asympt.}}{\leq} \frac{1}{2} |\partial^* \Delta|$$

Proof of Markov Theorem! Let $c_d = (2 \cdot 3^d)^{-1}$ fix $\Delta \subset \mathbb{Z}^d$ finite  connected with connected complement. Suppose A is ν -connected with $\bar{\Delta}^\nu = A$. ($|A| > \epsilon^{d/(d-1)} \geq (3\nu)^d$) (for the lemma).

then $|\Delta| \geq \tilde{N}^{-d} |\Lambda|$, standard isoperimetry on \mathbb{Z}^d

$$|\partial^* \Delta| \geq c_5 |\Delta|^{1-1/d} \geq c_5 \tilde{N}^{1-d} |\Lambda|^{1-1/d} \quad \text{set } c_2 = c_4 c_5 \tilde{N}^{1-d}$$

$$\{ |\partial^w \Lambda| < c_2 |\Lambda|^{d-1/d} \} \subset \{ |\partial^w \Lambda| < c_4 |\partial^* \Lambda| \}$$

$$\text{And } |\Lambda| \geq \epsilon^{d/(d-1)} \Rightarrow |\partial^* \Delta| \geq c_5 \tilde{N}^{1-d} \epsilon.$$

Since $\mathbb{P}(\mathbb{1}_{\mathcal{C}_2(\epsilon)})$ dominates dual percolation with $\eta_w(\epsilon) = 1 - \epsilon_N$

$$\begin{aligned} & \mathbb{P}(\exists \Delta \neq \emptyset, \text{w-connected}, |\Lambda| \geq \epsilon^{1/(d-1)}, \tilde{\Lambda}^w = \Delta, |\partial^w \Delta| < c_2 |\Lambda|^{d-1/d}) \\ & \leq \mathbb{P}(\sum_{x \in \partial^* \Delta} \mathbb{1}_{\mathcal{C}_2(\epsilon)} \leq \frac{1}{2} |\partial^* \Delta|) \leq 2 \frac{|\partial^* \Delta|}{\epsilon_N} \left(\frac{1}{2}\right) |\partial^* \Delta| \quad \text{union bound.} \end{aligned}$$

do we choose ϵ_r small enough s.t. ϵ_r^n is

smaller than ϵ of sets with boundary n .



RW on Percolation cluster on \mathbb{Z}^d

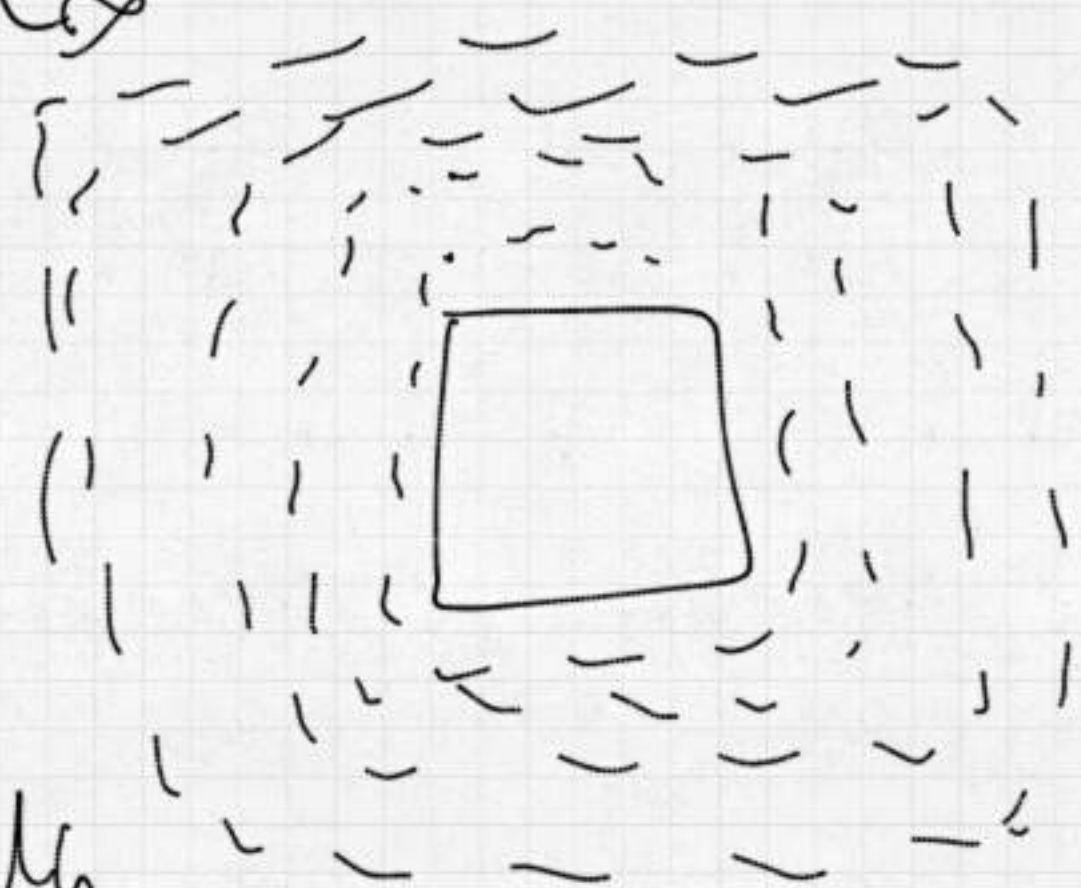
$$P_w [X_{n+1} = z \mid X_n = y] = \begin{cases} \frac{1}{2d} & (\|z-y\|_1 = 1, z \in C_y) \\ 1 - \frac{\deg_w(y)}{2d} & z=y \\ 0 & \text{otherwise} \end{cases}$$

Let $\phi(r) = \frac{|\partial S|}{|S|} : |S| \leq r$

then (Morris, Peres 2005)

It

$$h \geq 1 + \int_4^{4r\varepsilon} \frac{4dh}{h\phi^2(h)}$$



then $P_w^n(X_1 = y) \in \varepsilon$.

$$C = \frac{1}{h^{d-2}}$$

If $r \geq \sqrt{|S|} \geq h^{1/3}$ then $|\partial S| \geq c|S|^{d-1/d}$

$$\frac{|\partial S|}{|S|} \geq \frac{|\partial S|}{|S|^{d-1/d} |S|^{1/d}} \geq \frac{c}{|S|^{1/d}} \geq \frac{c}{r^{1/d}}$$

$$\int_{h^{1/3}}^{\sqrt{\epsilon}} \frac{dh}{h} = \int_{h^{1/3}}^{\sqrt{\epsilon}} \frac{1}{h} \frac{c}{h^{2/d}} dh \approx \frac{1}{\epsilon^{2/d}}$$

$$\Rightarrow \text{if } \epsilon = \frac{1}{N^{d/2}} \Rightarrow \rho^h(x_{ij}) \leq \frac{C}{N^{d/2}}$$

If $|S| < h^{1/3}$ then $\frac{|\partial S|}{|S|} \geq \frac{1}{h^{1/3}} \Rightarrow \int_1^{h^{1/3}} \frac{dh}{h} \leq h^{1/3} \cdot h^{2/3} = h$

transient for $d \geq 3$

$$P(X_h = (0, \dots, 0)) = 0$$

$$\text{since } \sum P^n(0,0) = \sum \frac{c}{n^{d/2}} \leq \infty$$

Invariance Principle: Consider

$$B_n(t) = \frac{1}{\sqrt{n}} \left[X_{\lfloor nt \rfloor} + (t_n - \lfloor nt \rfloor) (X_{\lfloor nt \rfloor + 1} - X_{\lfloor nt \rfloor}) \right]$$

$d \geq 2, \rho > \rho_c(d)$

Thm 1: the law of $(B_n(t) : 0 \leq t \leq 1)$ on $(C[0,1], W_T)$ converges

weakly to an isotropic Brownian motion with positive diffusion constant $(\sigma^2 d)$